Lab 7: Dynamical Systems & Stability

This lab involves no code, so just submit a PDF of your answers. You may still work in pairs.

Phase Plane Portraits of Linear Systems

For each of the linear systems below:

a. Determine the state matrix A, where $\dot{x} = Ax$.

b. Classify the equilibrium x = 0 by examining the eigenvalues of A. Compute the eigenvalues by hand – you can check your answers with a computer/calculator.

c. Use a phase plane plotter to plot phase plane portraits. If the eigenvectors of A are real, draw them on top of the corresponding phase plane portrait. Compute the eigenvectors by hand for problems 2 and 4, the rest use a computer program. On all plots, pick 4 points (not along the eigenvectors) and plot an arrow in the direction of the vector field $f = (f_1, f_2)' = (\dot{x_1}, \dot{x_2})$.

1. $$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_1 - x_2 \end{aligned}$$

2. $$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -3x_1 - 4x_2 \end{aligned}$$

3. $$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -4x_1 - 2x_2 \end{aligned}$$

4. $$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2x_1 + 3x_2 \end{aligned}$$

5. $$\begin{aligned} \dot{x_1} &= -x_1 + x_2 \\ \dot{x_2} &= -2x_1 + x_2 \end{aligned}$$

6. Describe how the eigenvectors provide information about the shape of the phase plane portrait.

Linearization of Nonlinear Systems

For each of the nonlinear systems below:

a. Identify the equilibrium point(s).

b. Compute the linear approximation at each equilibrium point.

c. Classify the behavior of the linearized system around each equilibrium point.

7. $$\begin{aligned} \dot{x_1} &= x_1 x_2 \\ \dot{x_2} &= -{x_1}^2 - x_2 \end{aligned}$$

8. $$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1 - 3x_2 + {x_1}^2 x_2 \end{aligned}$$

9. $$\begin{aligned} \dot{x_1} &= -{x_1}^3 - x_2 \\ \dot{x_2} &= 2x_1 - {x_2}^3 \end{aligned}$$

10. $$\begin{aligned} \dot{x_1} &= -x_1 - {x_2}^3 \\ \dot{x_2} &= -{x_1}^3 + x_2 \end{aligned}$$

11. $$\begin{aligned} \dot{x_1} &= -x_1(1 + {x_2}^2) \\ \dot{x_2} &= -x_1 + x_2 \end{aligned}$$

Stability through Lyapunov Functions

For each of the nonlinear systems below, investigate the stability of the equilibrium point (0,0)' using the Lyapunov function $V(x_1, x_2) = \frac{1}{2}x_1{}^2 + \frac{1}{2}x_2{}^2$. In each case specify whether the equilibrium point is stable, asymptotically stable, globally asymptotically stable, or unstable.

12. $$\begin{aligned} \dot{x_1} &= -x_1(1 + x_2{}^2) \\ \dot{x_2} &= -x_2 - x_1{}^2 x_2 \end{aligned}$$

13. $$\begin{aligned} \dot{x_1} &= x_1 x_2{}^2 - x_1{}^3 \\ \dot{x_2} &= -x_1{}^2 x_2 - x_2{}^3 \end{aligned}$$

14. $$\begin{aligned} \dot{x_1} &= x_1{}^2 x_2 + 2x_1 x_2{}^2 + x_1{}^3 \\ \dot{x_2} &= -x_1{}^3 + x_2{}^3 \end{aligned}$$