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Structural control of single-input rank one bilinear systems ☆

Supratim Ghosh [1], Justin Ruths

Singapore University of Technology and Design, Singapore

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ABSTRACT

A bilinear dynamical system can be used to represent the model of a network in which the state obeys linear dynamics and the input is the edge weight of certain controlled edges in the network. We present algebraic and graph-theoretic conditions for the structural controllability of a class of bilinear systems with a single control where the input matrix is rank one. Subsequently, we use these conditions, given a system state graph, to develop an algorithm to design the location of controlled edges (the input matrix) such that the system is structurally controllable.

1. Introduction

Recent studies have employed concepts from structural control in order to bring control theoretic analysis to large-scale complex networks. The rapid rise of computational capabilities and access to data in recent years has led to modeling many important systems – from intracellular biochemical pathways to the redesigned smart power grid – as networks (Newman, 2010). Understanding fundamental control properties is a key requirement to systematically studying and, ultimately, influencing these important systems. Classic control techniques, however, do not scale well to provide a feasible assessment of these properties. Structural control has proven to be a useful tool towards this goal.

Structural controllability is a generalization of classic controllability in which systems are analyzed based only on their structure, i.e., the existence or absence of a direct effect of one state on the change of another, and not the exact rate at which the states influence each other. Structural control is, therefore, "parameter free" in the sense that the analysis holds for all parameter values, except for specific pathological cases. This type of control is well-suited to analyze network systems by providing simplifications to make methods tractable and a set of tools that do not depend on exact parameter values, because such values are rarely known for most networks.

Conditions for structural controllability rely on classical control results, therefore, the analysis of networks has been limited to the case of linear dynamical systems, modeled as networks (Liu, Slotine, & Barabasi, 2011; Ruths & Ruths, 2014). While this body of work has already been able to provide revealing insights that connect network structures, such as the degree distribution, to control properties, more realistic models of these systems would permit deeper and more relevant analysis. In a network modeled by linear control dynamics, input signals are applied exogenously to specific nodes in the network, the influence of which is then able to control the entire network. This mechanism of influencing a network is applicable, for example, in resource networks (pipeline networks, power grids, and supply chains) where volume is injected or removed at nodes to manage demand, or food web networks where species can be bred and released or culled to achieve a population size (Dunne, Williams, & Martinez, 2002).

More often, however, this model falls short of how influence is achieved in a network. In a road network, tolls can be imposed on certain roads to alleviate traffic at specific points in the network. Similarly, biochemical networks are typically not controlled through direct injection of a protein, but instead by administering a drug that effects the rate at which that protein is produced naturally by the body (Marinissen & Gutkind, 2001).

A linear control model represents top-down control of a network whereas a bilinear model represents incentive-driven

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E-mail addresses: supratim_ghosh@sutd.edu.sg (S. Ghosh), justinruths@sutd.edu.sg (J. Ruths).

[1] Tel.: +65 8515 8625.

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control from within a network. Both schemes require global oversight, since controls are generated centrally, however, the top-down (linear) scheme effects the states of the system directly, and the incentive-driven (bilinear) scheme effects the states indirectly by throttling the natural interaction between two states.

In this work we study structural control of bilinear systems for the purpose of applying these methods to networks with a bilinear control structure. We cannot leverage the work by Boukhobza and Hamelin (2007) on structural observability of bilinear systems, however, because the nonlinearity of bilinear systems does not enter the observability criterion, making controllability a significantly harder problem. Because structural controllability results rely on classic control results, we have built this work on top of the most general algebraic bilinear controllability results, which are known for a class of bilinear systems which have a single control and such that the input matrix is rank one (Evans & Murthy, 1977; Goka, Tarn, & Zaborszky, 1973). One of the major contributions of this work is the collection of intuitive graphical conditions which will be more easily generalized to a broader class of bilinear systems. At the same time this class of systems is not without direct application. Most systems employing regulatory control schemes are driven by a single controlling source with a broadcasting (rank one) structure of interaction. For example, the Federal Reserve sets the national interest rate so as to achieve market stability in the network of banks and loaning agencies within the United States.

Most network systems are composed of similar agents (nodes), and through their interaction the system evolves. Unlike engineered systems, in which, for example, a pump and a valve in a pipeline system are clear actuation points, these network systems do not have pre-established points at which control should be applied. In the network setting we need to design the input connectivity given the structured system so that the system and input together are controllable, a relatively new type of problem we call control configuration design. Methods for control configuration design exist for linear structured systems and correspond to selecting the (fewest) nodes in the network to receive exogenous input (Murota, 2000). Control configuration design for bilinear systems is, to date, an open question and involves placing additional (in particular, the fewest) edges with controlled edge weights such that the overall system of fixed edges and controlled edges is controllable. In the context of social networks, for example, this would be equivalent to establishing, removing, strengthening, or weakening interconnections of trust/mistrust among people so that the opinion of the group as a whole can be influenced.

The contributions of this work are twofold: first, for discrete-time single-input rank one bilinear systems we develop equivalent algebraic and graph-theoretic results for checking structural controllability, and second, we design an efficient algorithm for control configuration design for this same class of bilinear systems. Preliminary versions of these results were published in Ghosh and Ruths (2014a,b).

2. Background

We consider single-input homogeneous (without a linear control term) bilinear systems such that the input matrix is rank one. The state equation of the system is given by

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + u(t)\mathbf{B}\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state of the system and $u(t) \in \mathbb{R}$ denotes the control input to the system at time $t \in \mathbb{N}_0$; $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ denote the state and input matrices, respectively. Because we consider input matrices of rank one, the matrix $\mathbf{B}$ can be written as $\mathbf{B} = \mathbf{c}\mathbf{h}^{\mathrm{T}}$ where $\mathbf{c}, \mathbf{h} \in \mathbb{R}^n$. Thus, an alternate description of the state equation is

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + u(t)\mathbf{c}\mathbf{h}^{\mathrm{T}}\mathbf{x}(t). \tag{2}$$

Although select results exist for seemingly broader classes of bilinear systems, namely for controllability of multi-input and inhomogeneous systems, all of these results put highly restrictive assumptions on the form of the system matrices and are thus less general and interpretable (Evans & Murthy, 1978; Hollis & Murthy, 1981; Tie, Cai, & Lin, 2011). Even though the rank one condition on $\mathbf{B}$ is restrictive, a number of important classes of systems satisfy this requirement. One such example is the class of bilinear strict-feedback systems which are a class of nonlinear strict feedback systems (Khalil, 2002). For example, a strict feedback system of order 3 can be described by the following set of equations

$$\begin{aligned}
x_1(t+1) &= f_1\left(x_1(t)\right) + \gamma_1 x_2(t),\\
x_2(t+1) &= f_2\left(x_1(t), x_2(t)\right) + \gamma_2 x_3(t),\\
x_3(t+1) &= f_3\left(x_1(t), x_2(t), x_3(t)\right) + g_3\left(x_1(t), x_2(t), x_3(t)\right)u(t),
\end{aligned}$$

where the functions $f_i(\cdot)$ (with $i = 1, 2, 3$) and $g_3(\cdot)$ are linear in their variables; $\gamma_1, \gamma_2 \neq 0$. The overall state equation of the system is then given by (1) where $\mathbf{B}$ has all rows, except the last one, as zero rows. Another application of a single-input discrete-time rank-one bilinear system can be found in the context of wavelength-division multiplexing (Ishio, Minowa, & Noshu, 1984). The network consists of three parts: a multiplexer (that works according to the sparsity of $\mathbf{h}$), an amplifier with gain $u(t)$ and a demultiplexer (which works according to the structure of $\mathbf{c}$). Such networks appear the long-haul transmission where the sensors and actuators are far from each other and there is a bandwidth constraint of transmission and reception of data.

2.1. Structured systems

The notion of structured systems was introduced so that system properties could be evaluated and studied for systems that had a particular structure, regardless of the exact parameter values.

Mathematically the *structure* of structured systems is captured by matrix entries that are either fixed at zero (i.e., two states are known to have no direct interaction) or allowed to vary independently (i.e., the rate of the interaction between two states is given by an independent parameter). Therefore, in the structured version of (2) the structured matrices $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$ have entries that are either identically zero (denoted simply as 0) or free, able to take on any real number (denoted by λ_i or simply by $*$). An example of such a structured system is

$$\mathbf{x}(t+1) = \underbrace{\begin{bmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(t) + u(t) \underbrace{\begin{bmatrix} 0 \\ \lambda_3 \end{bmatrix}}_{\mathbf{c}} \underbrace{\begin{bmatrix} \lambda_4 & \lambda_5 \end{bmatrix}}_{\mathbf{h}^{\mathrm{T}}} \mathbf{x}(t),$$

where $\lambda_i \in \mathbb{R}$ for $i \in \{1, \ldots, 5\}$ is an independent parameter. We study the properties of these systems in a generic sense; i.e., the properties under consideration must hold for almost every choice of these free parameters. We will define this notion in terms of polynomials and algebraic varieties.

An algebraic variety is the zero set of a finite set of polynomials. An algebraic variety $V \subset \mathbb{R}^N$ is called a proper variety if $V \neq \mathbb{R}^N$ and nontrivial if $V \neq \emptyset$. A proper variety is one of the standard sets known to have Lebesgue measure zero (Polderman & Willems, 1998)

Definition 1. A property (e.g., controllability) is said to hold generically for a structured system if the set of values of the free parameters for which the property does not hold forms a proper algebraic variety.

One of the key advantages of studying structured systems is that analyzing generic properties tends to be much simpler than their classical (nonstructured) counterparts. This simplification makes studying the control related properties of large scale systems, like complex networks, tractable.

Definition 2. A structured system (2) described by $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ is said to be structurally equivalent to another system $(\bar{\mathbf{A}}, \bar{\mathbf{c}}, \bar{\mathbf{h}}^{\mathrm{T}})$ if there is a one-to-one correspondence between the locations of fixed zero and free entries of $\mathbf{A}$ and $\bar{\mathbf{A}}$, $\mathbf{c}$ and $\bar{\mathbf{c}}$, and $\mathbf{h}$ and $\bar{\mathbf{h}}$, respectively.

Definition 3. A structured system (2) described by $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ is said to be structurally controllable if there exists a nonstructured triple $(\bar{\mathbf{A}}, \bar{\mathbf{c}}, \bar{\mathbf{h}}^{\mathrm{T}})$ structurally equivalent to $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ and is controllable using the classic definition.

2.2. Graphical model

The motivation for this work originates from the network interpretation of bilinear systems. We find that this network representation not only provides an application for this theory but also greatly simplifies and facilitates the development and intuition of the theory itself.

In classic (nonstructured) linear and bilinear systems the state matrix $\mathbf{A}$ can be interpreted as the adjacency matrix of a network in which there is a connection from node (state) x_i to node (state) x_j if $\mathbf{A}_{ji} \neq 0$. More specifically, the value $\mathbf{A}_{ji}$ denotes the weight of the edge that connects x_i to x_j. In linear systems the input matrix encodes the connections from a set of exogenous control signals to the state nodes. In bilinear systems, however, the input matrix $\mathbf{B}$ encodes connections from one state node (x_i) to another (x_j) that have edge weights that can be controlled, i.e., edge weights given by $\mathbf{B}_{ji}u(t)$.

In structured systems analysis, the dependence of system properties on the values of the parameters (i.e., the weights of the edge connectivities) is removed. Therefore, the network representation is even more appropriate in this context because the structure of the connectivity is the only relevant factor.

To facilitate our analysis, we introduce a second graphical model motivated by the $\mathbf{B} = \mathbf{ch}^{\mathrm{T}}$ decomposition used in (2) and capable of being analyzed as a structured linear system (Dion, Commault, & van der Woude, 2003; Lin, 1974; Reinschke, 1988). We first define a linear system *associated* to the bilinear system (2), following Goka et al. (1973),

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{c}\tilde{u}(t) \qquad (3)$$
$$\tilde{y}(t) = \mathbf{h}^{\mathrm{T}}\mathbf{x}(t)$$

where $\tilde{u}(t)$ and $\tilde{y}(t)$ denote the pseudo-input and pseudo-observation, respectively.

The directed graph $G = (\mathcal{V}, \mathcal{E})$ comprises a vertex set $\mathcal{V}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. In the context of the associated linear system described by (3), the vertex set $\mathcal{V}$ is the union of three sets: $\mathcal{X} = \{x_1, \dots, x_n\}$ which denotes the state vertices; $\mathcal{U} = \{\tilde{u}\}$ which denotes the pseudo-input; and $\mathcal{Y} = \{\tilde{y}\}$ which denotes the pseudo-observation. The edge set $\mathcal{E}$ is defined as $\mathcal{E} = \mathcal{E}_{\mathbf{A}} \cup \mathcal{E}_{\mathbf{c}} \cup \mathcal{E}_{\mathbf{h}}$, where $(x_i, x_j) \in \mathcal{E}_{\mathbf{A}}$, meaning there is an edge from x_i to x_j, if and only if $\mathbf{A}_{ji} \not\equiv 0$; $(\tilde{u}, x_j) \in \mathcal{E}_{\mathbf{c}}$ if and only if $\mathbf{c}_j \not\equiv 0$; and $(x_i, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}$ if and only if $\mathbf{h}_i \not\equiv 0$ (we use $\not\equiv 0$ to denote that it is a free parameter and not a fixed zero value). The original bilinear system (1) has a *controlled edge* from x_i to x_j, $(x_i, x_j) \in \mathcal{E}_{\mathbf{B}}$, if and only if $\mathbf{B}_{ji} = \mathbf{c}_j\mathbf{h}_i \not\equiv 0$. If (x_i, x_i) is in $\mathcal{E}_{\mathbf{A}}$ or $\mathcal{E}_{\mathbf{B}}$ it is called a self-loop or controlled self-loop, respectively.

A large part of the simplification offered by structured systems comes from the fact that matrix products simplify to walks on the corresponding graph. A walk W on a graph is a sequence of nodes and edges such that the end vertex of a preceding edge is the begin vertex of the next. The length of a walk W is equal to the number of edges present in the walk. A ℓ-length walk $W = \{(w_0, w_1), \dots, (w_{\ell-1}, w_\ell)\}$ is often represented as $w_0 \to w_1 \to \cdots \to w_\ell$. A walk is a *path* if it does not contain any repeated vertices. A walk W is said to be *closed* if the begin and end vertex are the same. A closed path is called a cycle. A path is called $\mathcal{U}$-rooted ($\mathcal{Y}$-topped) if its begin (end) vertex is $\tilde{u}$ ($\tilde{y}$). A number of mutually disjoint $\mathcal{U}$-rooted ($\mathcal{Y}$-topped) paths is called a $\mathcal{U}$-rooted ($\mathcal{Y}$-topped) path family. Similarly, a set of disjoint cycles is called a cycle family. A collection of walks is said to *cover* all the vertices in $\mathcal{X}$ if every vertex in $\mathcal{X}$ belongs to at least one of the walks.

2.2.1. Cacti

The community has introduced the notion of a cactus of a directed graph to operationalize the notions of the $\mathcal{U}$-rooted or $\mathcal{Y}$-topped path/cycle families (Dion et al., 2003; Lin, 1974). The cactus is a minimal subgraph that retains the control-relevant edges of the underlying linear dynamical system. The cactus representation of a directed graph effectively assures that every state is reachable from a control (or analogously that every state can reach the observation) and that the Kalman rank condition is satisfied.

A cactus consists of at most one *stem* (a $\mathcal{U}$-rooted or $\mathcal{Y}$-topped path) and any number of *buds*, which are cycles that are connected to from either the stem or from other buds via a *distinguished edge*. Several *cacti*, a collection of mutually disjoint cactus subgraphs, may be required to cover the entire graph. The cacti representation of a directed graph can be obtained in polynomial time using the maximum matching algorithm (Dion et al., 2003; Hopcroft & Karp, 1973). The maximum matching of a directed graph produces a potentially non-unique collection of paths and cycles that cover the entire graph from which we can build the cacti (note that the distinguished edges connecting the buds to other cacti components are not contained in the matching).

A cactus is by definition a minimal, *dilation*-free network structure in which there are no inaccessible nodes (Lin, 1974). The notion of a dilation is the graphical analog of the generic rank condition.

Definition 4. A graph $G = (\mathcal{V}, \mathcal{E})$ is said to possess a dilation if there exists a set $S \subset \mathcal{X}$ of nodes such that $|T(S)| < |S|$ where $T(S) = \{x_i : (x_i, x_j) \in \mathcal{E}, x_j \in S\}$; i.e., $T(S)$ denotes the set of nodes with edges into S.

Remark 5. A dilation can be intuitively described as an expansion in the network, where the set $S \subset \mathcal{X}$ is composed only of state nodes, not control nodes. Constructing cacti for the directed graph can be understood as the process of adding control (observation) nodes and edges from (to) these control (observation) nodes in order that all such expansion points occur at the control nodes.

Remark 6. In the case of a single dilation, there can be at most one stem, therefore, a single cactus can cover the network. The implication is, however, one-sided since, for example, if $\mathbf{A}$ is a structured diagonal matrix where each node has a self-loop, then no stems are present and, therefore, there are no dilations in $G(\mathbf{A})$.

2.3. Objectives

This paper addresses the structural controllability of single-input rank one discrete time bilinear systems of form (2) by considering the following problems:

(1) *Analysis*: given structured matrices $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$, identify algebraic and graph-theoretic conditions that completely describe the controllability of (2), and
(2) *Control configuration design*: given a structured matrix $\mathbf{A}$, identify $\mathbf{c}$ and $\mathbf{h}$ with minimum number of nonzero entries such that $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ is structurally controllable.

3. Analysis of structural controllability

Our exact conditions for controllability of single-input rank one discrete time bilinear systems depend upon two results: controllability of structured linear systems and of classic (nonstructured) bilinear systems.

3.1. Structural controllability of linear systems

A structured linear system is described by

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4a}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{4b}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^p$ denote the state, input, and measurements of the system, respectively. The matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are structured matrices such that only the location of zero and nonzero entries are known.

Definition 7. The generic rank of a matrix $\mathbf{M}$ is the maximal rank that can be obtained by fixing the free entries of $\mathbf{M}$.

Definition 8. A structured matrix pair $(\mathbf{A}, \mathbf{B})$ of compatible dimensions is said to be reducible if and only if there exists a permutation matrix $\mathbf{P}$ such that the pair $(\mathbf{A}, \mathbf{B})$ satisfies

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \text{and} \quad \mathbf{P}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}.$$

A reducible system has states that cannot be reached by the control either directly or indirectly, identified visually by this partitioning of the states.

The following result establishes algebraic and graph-theoretic equivalent conditions for structural controllability (and observability) of linear systems (Dion et al., 2003; Glover & Silverman, 1976; Lin, 1974; Reinschke, 1988; Shields & Pearson, 1976).

Lemma 9. *For the linear system (4a) described by the structured pair $(\mathbf{A}, \mathbf{B})$, the following are equivalent:*

1. *The system (4a) is structurally controllable.*
2. *The generic rank of $[\mathbf{A}\,\mathbf{B}]$ equals n and the pair $(\mathbf{A}, \mathbf{B})$ is irreducible.*
3. *The graph $G(\mathbf{A}, \mathbf{B})$ is covered by a disjoint union of a $\mathcal{U}$-rooted path and cycle families.*
4. *The graph $G(\mathbf{A}, \mathbf{B})$ is covered by a disjoint union of cacti.*

We define $G(\mathbf{A}, \mathbf{B})$ as the graph $G(\mathbf{A})$ augmented by adding nodes corresponding to each of the independent controls and connecting them to the state nodes by the edges indicated in the input matrix $\mathbf{B}$. Similar results hold for structural observability of (4) with $G(\mathbf{A}, \mathbf{B})$ replaced by $G(\mathbf{A}, \mathbf{C})$ and $\mathcal{U}$-rooted replaced by $\mathcal{Y}$-topped.

3.2. Controllability of bilinear systems

By virtue of the inherent multiplication of control and state terms, the notion of (nonstructured) controllability of bilinear systems is slightly modified from the linear case, namely by omitting the origin (Evans & Murthy, 1977; Goka et al., 1973).

Definition 10. The bilinear system (2) described by $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$ is said to be controllable on $\mathbb{R}^n \setminus \{0\}$, if given any initial state $\mathbf{x}_0 \in \mathbb{R}^n \setminus \{0\}$ and a final desired state $\mathbf{x}_f \in \mathbb{R}^n \setminus \{0\}$, one can design a sequence of control inputs $u(0), \ldots, u(T-1)$ such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_f$ for some $T \in \mathbb{N}_0$.

The most general and approachable algebraic results for (nonstructured) controllability of bilinear systems exist only for the class of single input, rank one systems described by (2) (Evans & Murthy, 1977; Funahashi, 1979; Goka et al., 1973).

Lemma 11. *The bilinear system (2) described by $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$ is controllable if and only if the following hold:*

1. $\operatorname{rank}\begin{bmatrix}\mathbf{c} & \cdots & \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{h} & \cdots & (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{h}\end{bmatrix}^{\mathrm{T}} = n$,
2. *The greatest common divisor (g.c.d.) of I equals one where $I \triangleq \{j : \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \neq 0, j = 1, \ldots, 2n\}$.*

3.3. Structural controllability of bilinear systems

Our main result states the controllability conditions for bilinear systems (2). We define $\mathcal{X}_{\mathcal{Y}}$ as the collection of nodes in $\mathcal{X}$ that have a direct edge to $\tilde{y}$ in $G(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$; i.e., the set of nodes $x_i \in \mathcal{X}$ such that $(x_i, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}$. Define $\mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$ as the set of all (possibly self-intersecting) walks from $\tilde{u}$ to any vertex in $\mathcal{X}_{\mathcal{Y}}$ of length at most $2n$.

Theorem 12. *For the bilinear system (2) described by the structured triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$, the following are equivalent:*

1. *The system (2) is structurally controllable.*

2a. *The generic rank of $\mathbf{M} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{h}^{\mathrm{T}} & * \end{bmatrix}$ equals $n+1$; $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are irreducible.*

b. *The greatest common divisor (g.c.d.) of I equals one where $I \triangleq \{j : \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0, j = 1, \ldots, 2n\}$.*

3a. *$G(\mathbf{A}, \mathbf{c})$ and $G(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ are respectively covered by a disjoint union of a $\mathcal{U}$-rooted and $\mathcal{Y}$-topped path and a cycle family.*

b. *There exists a collection of walks of coprime lengths in $\mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$.*

Theorem 12 characterizes the structural controllability of single-input, rank one bilinear systems in terms of algebraic (part 2) and graph-theoretic (part 3) conditions. The structural controllability of bilinear systems depends on two factors: the controllability and observability of the associated linear system (3), which is given by parts 2a or 3a, and the existence of a coprimeness property, which is given by part 2b or 3b.

Remark 13. One of the implications of the condition 2a is that $\mathbf{A}$ must have generic rank of at least $n-1$. This implies by way of Remark 6, which says that $G(\mathbf{A})$ has a single dilation if $\mathbf{A}$ has rank $n-1$, that a single cactus covers the graph. This constraint is imposed because the system has a single input. To achieve controllability each stem must receive input from a different control. Since (2) has only one independent control, there can be at most one stem and so one cactus.

Part 3a is also equivalent to the following.

3a′. *The graphs $G(\mathbf{A}, \mathbf{c})$ and $G(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ are spanned by a cactus rooted in $\mathcal{U}$ and topped by $\mathcal{Y}$, respectively.*

Since the linear algebra (2a, 2b) and the graphical (3a′, 3b) conditions are equivalent (with comparable computational complexities), the primary advantage of the graphical approach is that it provides greater intuition to the problem (thereby more easily generalized) and that networks provide a more natural representation of the system.

Remark 14. Checking the controllability and observability is dominated either by checking the rank of a matrix (2a), which (Bunch & Hopcroft, 1974) identifies as equivalent to matrix multiplication, $O(n^3)$, or by the maximum matching (3a′), where the (Hopcroft & Karp, 1973) algorithm runs in $O(L\sqrt{n})$, $L = |\mathcal{E}|$ is the number of edges in the graph. Improvements on matrix multiplication can yield complexities lower than $O(n^{2.376})$ and maximum matching on sparse graphs run in less than $O(n^{2.5})$ (Coppersmith & Winograd, 1990). Checking for coprime paths uses $4n$ matrix–vector products, leading to a complexity bounded by $O(n^3)$. A direct graphical method for determining the existence of coprime paths in a network is part of our ongoing work.

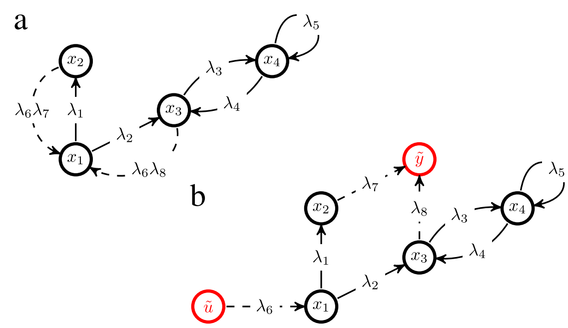

Fig. 1. (a) The directed graph of the four-state example bilinear system and (b) the directed graph corresponding to the associated linear system, with pseudo-input $\tilde{u}$ and pseudo-output $\tilde{y}$.

A special case of this result exists if the state matrix $\mathbf{A}$ has generic rank n and the corresponding graph $G(\mathbf{A})$ is strongly connected. In such a situation, the presence of just a single self-loop in the graph (i.e., $\mathbf{A}_{ii} \not\equiv 0$ for some $i \in \{1, \ldots, n\}$) grants that the triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ is structurally controllable for any non-zero $\mathbf{c}$ and $\mathbf{h}$ (if $\mathbf{c}$ or $\mathbf{h}$ is zero, then that would imply $\mathbf{B} = \mathbf{0}$ which would make the system uncontrollable for any input).

Proposition 15. *Suppose $G(\mathbf{A})$ is strongly connected, $\mathbf{A}$ is of generic rank n, and $(x_i, x_i) \in \mathcal{E}_{\mathbf{A}}$ for some $i \in \{1, \ldots, n\}$. Then the bilinear system described by (2) is structurally controllable for any structured $\mathbf{c}$ and $\mathbf{h}$ such that $\mathbf{B} = \mathbf{c}\mathbf{h}^{\mathrm{T}} \not\equiv \mathbf{0}$.*

If in addition $G(\mathbf{A})$ has self-loops on all nodes, i.e., $\mathbf{A}_{ii} \not\equiv 0$ for all $i \in \{1, \ldots, n\}$, then $\mathbf{A}$ automatically has full generic rank. This proposition automatically applies to connected undirected (bidirectional) graphs, since every connected undirected graph is strongly connected. Examples of such graphs occur frequently in multi-agent consensus problems (Jadbabaie, Lin, & Morse, 2003; Xiao & Boyd, 2004).

3.4. Example

To help illustrate the important aspects of Theorem 12, consider the four-state bilinear system in Fig. 1:

$$\mathbf{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & \lambda_4 \\ 0 & 0 & \lambda_3 & \lambda_5 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(t) + u(t) \underbrace{\begin{bmatrix} \lambda_6 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{c}} \underbrace{\begin{bmatrix} 0 & \lambda_7 & \lambda_8 & 0 \end{bmatrix}}_{\mathbf{h}^{\mathrm{T}}} \mathbf{x}(t),$$

such that the parameters $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, 8$. The graphical representation of the associated linear system is also shown in Fig. 1 and given by $G = (\mathcal{V}, \mathcal{E})$ with the vertex set $\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}$ with $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, $\mathcal{U} = \{\tilde{u}\}$, and $\mathcal{Y} = \{\tilde{y}\}$. The edge set $\mathcal{E}$ is the union of the edge sets $\mathcal{E}_{\mathbf{A}} = \{(x_1, x_2), (x_1, x_3), (x_3, x_4), (x_4, x_3), (x_4, x_4)\}$, $\mathcal{E}_{\mathbf{c}} = \{(\tilde{u}, x_1)\}$, and $\mathcal{E}_{\mathbf{h}} = \{(x_2, \tilde{y}), (x_3, \tilde{y})\}$. It can be easily seen that $G(\mathbf{A})$ has one self-loop on node x_4. The structures of $\mathbf{c}$ and $\mathbf{h}$ indicate there are two controlled edges in the network (the dashed lines in Fig. 1).

First, note that the generic rank of $\mathbf{A}$ equals 3 which is equal to $n - 1$. However the structures of $\mathbf{c}$ and $\mathbf{h}$ guarantee that $\mathbf{M}$ has full generic row rank. Furthermore, both $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are irreducible which tells us that $(\mathbf{A}, \mathbf{c})$ is structurally controllable and $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is structurally observable. Alternatively, one can deduce this from the directed graph G. For example, $G(\mathbf{A}, \mathbf{c})$ contains a $\mathcal{U}$-rooted cactus, which includes all edges except the self-loop on x_4. Similarly, $G(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ can be shown to contain a $\mathcal{Y}$-topped cactus, including all edges except (x_1, x_3) and (x_4, x_4). These clearly satisfy the conditions listed in part 3a′ of Theorem 12.

Similarly it can be observed that $\mathbf{h}^{\mathrm{T}}\mathbf{c} = 0$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}\mathbf{c} = (\lambda_1\lambda_7 + \lambda_2\lambda_8)\lambda_6 \not\equiv 0$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}^2\mathbf{c} = 0$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}^3\mathbf{c} = \lambda_2\lambda_3\lambda_4\lambda_6\lambda_8 \not\equiv 0$ and $\mathbf{h}^{\mathrm{T}}\mathbf{A}^4\mathbf{c} = \lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_8 \not\equiv 0$. Thus, $2, 4, 5 \in I$ and hence $\gcd(I) = 1$, which satisfies the condition 3b of Theorem 12. Alternatively, this can be deduced from the fact that there exist walks $W_1 = \tilde{u} \to x_1 \to x_2$ and $W_2 = \tilde{u} \to x_1 \to x_3$ of length 2, $W_3 = \tilde{u} \to x_1 \to x_3 \to x_4 \to x_3$ of length 4, and $W_4 = \tilde{u} \to x_1 \to x_3 \to x_4 \to x_4 \to x_3$ of length 5 with $x_2, x_3 \in \mathcal{X}_{\mathcal{Y}}$, which have coprime lengths. Since the algebraic and graphical conditions listed in Theorem 12 are satisfied, this system is structurally controllable. For instance, consider the realization in which $\boldsymbol{\lambda} = \mathbf{1}$. It can be easily seen that $(\mathbf{A}, \mathbf{c})$ is controllable and $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ observable since the both the associated matrices have full rank. Furthermore, $\mathbf{h}^{\mathrm{T}}\mathbf{c} = 0$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}\mathbf{c} = 2$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}^2\mathbf{c} = 0$, and $\mathbf{h}^{\mathrm{T}}\mathbf{A}^3\mathbf{c} = \mathbf{h}^{\mathrm{T}}\mathbf{A}^4\mathbf{c} = 1$ imply that $2, 4, 5 \in I$ and thus, $\gcd(I) = 1$.

If the self-loop (x_4, x_4) is removed, the associated linear system is still both structurally controllable and observable. However, the set I now contains walks of only even lengths (e.g., $2, 4, \ldots$), which are not coprime to each other. Therefore, part 2b (or part 3b) of Theorem 12 fails and the bilinear system is not structurally controllable.

4. Control configuration design

In the previous section we developed the exact conditions required for structural controllability of single-input, rank one bilinear systems. Many natural systems, however, especially in the context of large networks, do not automatically have an associated set of controls with known connectivity. In this section, we now use our controllability conditions to devise an algorithm to generate the placement of a minimum number of edges that must be added to guarantee structural controllability. We call this procedure *control configuration design* for single-input, rank one bilinear systems described in (2).

Given the sparsity structure of $\mathbf{A}$ (locations of fixed interconnections between the states/nodes), our objective is to design $\mathbf{c}$ and $\mathbf{h}$ with maximum sparsity (having maximum zeros, or equivalently, the fewest controlled edges) so that the overall system described by $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ is structurally controllable. Recall that a connection $\tilde{u}$ to x_j implies that $\mathbf{c}_j \not\equiv 0$; i.e., $\mathbf{c}_j$ is set to be a free parameter. Similarly, connecting x_i to $\tilde{y}$ is equivalent to making $\mathbf{h}_i \not\equiv 0$. Selecting the nonzero entries in $\mathbf{c}$ and $\mathbf{h}$ is equivalent to constructing the cactus for $G(\mathbf{A}, \mathbf{B})$ and guaranteeing the coprimeness condition. We are aided in this goal by the fact, established in Remark 13, that single-input bilinear systems must be covered by a single cactus, i.e., one stem and any number of buds.

The following procedure formalizes the process of assigning the attachment of cycles in the matching to the final cactus structure. As we will see, the number of controlled edges in the final control configuration is minimized by aggregating together as many cycles as possible.

(1) Obtain a maximum matching for the graph $G(\mathbf{A})$, which in this case is composed of at most one stem with any number of cycles. Let $\mathcal{M}_{\mathbf{A}}$ denote the collection containing the stem and cycles given by the matching.
(2) Create a new graph of matching components (stem and cycles) $G(\mathcal{M}_{\mathbf{A}})$, which collapses each matching component $\mathcal{M}_i$ into a single node m_i. An edge exists from m_i to m_j if there exists an edge from one of the nodes in $\mathcal{M}_i$ to one of the nodes in $\mathcal{M}_j$ in the original graph $G(\mathbf{A})$.

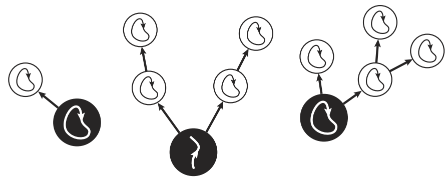

Fig. 2. A schematic of the spanning forest $G(\mathcal{M}_{\mathbf{A}})$ or $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$, where matching components (stem and cycles) of $G(\mathbf{A})$ or $G(\mathbf{A}^{\mathrm{T}})$ are collapsed into nodes. The black-filled nodes are the roots of the spanning forest. If a stem is present in the matching it must be the root of one of the trees of the forest. This forest has 3 trees.

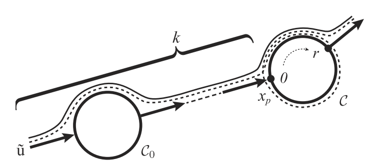

Fig. 3. The coprimeness condition is satisfied by selecting the appropriate rth node in the cycle and winding around the bud $\mathcal{C} \in \mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$ once.

(3) Find a spanning forest of $G(\mathcal{M}_{\mathbf{A}})$ with the minimum number of trees and with the stem (if one exists) as the root of one of the trees. Effectively this spanning forest is meant to determine the minimum number of components in $G(\mathcal{M}_{\mathbf{A}})$ by selecting the appropriate distinguished edges with which to connect buds to the stem and to other cycles; each of these components will require a controlled edge.
(4) We now formalize our final notation: $\mathcal{S}_{\mathbf{A}}$ is the single stem, located at the root of one of the trees; $\mathcal{R}_{\mathbf{A}}$ are the cycles located at the root of the rest of the trees; and $\mathcal{B}_{\mathbf{A}}$ are the rest of the cycles. A diagram of a simple such spanning forest is shown in Fig. 2.
(5) Although we could calculate the matching directly for the reverse graph $G(\mathbf{A}^{\mathrm{T}})$, the single cacti constraint forces $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}} = \mathcal{S}_{\mathbf{A}}$ (with the edge orientation reversed). Cycles of the matching of $G(\mathbf{A})$, therefore, will still be cycles in the matching of $G(\mathbf{A}^{\mathrm{T}})$, thus $\mathcal{M}_{\mathbf{A}^{\mathrm{T}}} = \mathcal{M}_{\mathbf{A}}$. However, because the edge orientation is reversed, $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$ is different and so the spanning forest with a minimum number of trees is also different. As before we identify $\mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$ and $\mathcal{B}_{\mathbf{A}^{\mathrm{T}}}$ from the spanning forest of $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$.

With these sets defined, we now identify the placement of the controlled edges to complete the control configuration design algorithm.

(6) If the cactus of $G(\mathbf{A})$ has a stem, let its length be $\ell - 1$ and connect $\tilde{u}$ to the base node (say x_j) and $\tilde{y}$ from the top node (say x_i) of the stem, i.e., set $\mathbf{c}_j$ and $\mathbf{h}_i$ to be free parameters. Ultimately, to satisfy the covering condition (all nodes can be reached by the control and can reach the observation) each tree in the two spanning forests mentioned above must be connected to the control and observation. However, to also satisfy the coprimeness condition, we proceed according to one of the following cases:

I. **Spanning forests both have exactly one tree, with stem as the root**: Since all nodes are contained in the stem and the cycles connected from the stem, connecting $\tilde{u}$ and $\tilde{y}$ as above guarantees that all nodes are reachable from $\tilde{u}$ and can reach $\tilde{y}$. To check the coprime walk condition, let $I = \{k\colon \mathbf{A}_{ij}^{k-1} \not\equiv 0, k = 1, \ldots, 2n\}$. If $\gcd(I) = 1$, then the condition is naturally satisfied. Otherwise add a single extra edge from the bottom node of the stem, x_j, to $\tilde{y}$. This inserts a controlled self-loop at x_j, $\tilde{u} \to x_j \to \tilde{y}$, which satisfies the coprime walk condition since $\gcd(\ell, 1) = 1$ (ℓ is the length of the walk from $\tilde{u}$ to the neighbor set of $\tilde{y}$, $\mathcal{X}_{\mathcal{Y}}$).

Note that this procedure also addresses the case when there is only a stem and no cycles.

II. **Common root cycle in spanning forests**: If there is at least one cycle that is a root in both spanning forests, i.e., $\mathcal{C} \in \mathcal{R}_{\mathbf{A}}$ and $\mathcal{C} \in \mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$, choose any node (say x_k) of such a cycle and connect $\tilde{u}$ to x_k and x_k to $\tilde{y}$. This adds a controlled self-loop in the system which again satisfies the gcd condition.

To satisfy the covering condition, also connect $\tilde{u}$ to any node in each of the rest of the root cycles $\mathcal{R}_{\mathbf{A}}$ and connect $\tilde{y}$ from any node in each of the rest of the root cycles in $\mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$.

If there is no stem in the matching but a cycle is a root in both spanning forests, this case applies without modification.

III. **No common root cycle in spanning forests**: Select a cycle $\mathcal{C} \in \mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$ that is a root in the spanning forest of $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$, but not a root in the spanning forest of $G(\mathcal{M}_{\mathbf{A}})$, $\mathcal{C} \in \mathcal{B}_{\mathbf{A}}$. Let $\mathcal{C}_0 \in \mathcal{R}_{\mathbf{A}}$ be the root of the tree that contains $\mathcal{C}$. Connect the control $\tilde{u}$ to any node in the cycle $\mathcal{C}_0$, let k be the distance from $\tilde{u}$ to the end vertex (say $x_p \in \mathcal{C}$) of the distinguished edge connecting to $\mathcal{C}$, and let c be the length of the cycle $\mathcal{C}$.

A simple algebraic exercise reveals the fact that $\gcd(k + r, k+r+c) = 1$ coincides with $\gcd(k+r, c) = 1$. Furthermore, an inductive argument shows that there always exists a $r \in \{0, \ldots, c - 1\}$ such that $\gcd(k + r, c) = 1$, which states that in any interval of c integers there is a number that is coprime with c. In the context of this problem, these facts taken together prove that the rth node in the cycle $\mathcal{C}$ (starting from x_p as the 0th node) can be selected such that the length of the walk from $\tilde{u}$ to x_p to the rth node $(k + r)$ is coprime to the length of the walk from $\tilde{u}$ to x_p, around the cycle once, and then to the rth node $(k + c + r)$. Therefore, by connecting the rth node to $\tilde{y}$ we guarantee the coprime path condition. This is diagramed in Fig. 3.

To satisfy the covering condition, also connect $\tilde{u}$ to any node in each of the rest of the root cycles $\mathcal{R}_{\mathbf{A}}$ and connect $\tilde{y}$ from any node in each of the rest of the root cycles in $\mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$.

If there is no stem in the matching and there is no cycle that is a common root in both spanning forests, this case applies without modification.

The case where $\mathcal{C}$ is a root in the spanning forest of $G(\mathcal{M}_{\mathbf{A}})$, $\mathcal{C} \in \mathcal{R}_{\mathbf{A}}$, but is not a root in the spanning forest of $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$, $\mathcal{C} \in \mathcal{B}_{\mathbf{A}^{\mathrm{T}}}$, is treated identically, however, the roles of $\tilde{u}$ and $\tilde{y}$ are reversed.

Our goal is to guarantee two sets of conditions: controllability and observability of the associated linear system and the coprimeness condition. The former is done by connecting $\tilde{u}$ to and $\tilde{y}$ from the roots of the respective spanning forests. Assume that the number of trees in the spanning forest of $G(\mathcal{M}_{\mathbf{A}})$ is $n_{\mathbf{c}}$ and that in the spanning forest of $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$ is $n_{\mathbf{h}}$. In other words, at minimum $n_{\mathbf{c}}$ interconnections are required from $\tilde{u}$ ($n_{\mathbf{h}}$ to $\tilde{y}$) to guarantee the controllability (and observability) of the associated linear system. Thus, the minimum number of controlled edges required to guarantee the controllability of the overall bilinear system would be $n_{\mathbf{c}}n_{\mathbf{h}}$ (or equivalently, the number of non-zero entries in $\mathbf{B}$). What we find is that bilinear systems are relatively easy to control, in the sense that most of the cases above attain this minimum even when the coprimeness condition is tested. The exception is Case I, in which only a single extra controlled edge (a self edge) may be required to satisfy the coprime walks. Note that for Case I the number of required controlled edges is either 1 or 2, so even in this case the coprimeness condition does not cause significant effect on the size of the control configuration.

Remark 16. Self-loops can play a dramatically simplifying role in the control of bilinear systems. Once the conditions of controllability and observability of the associated linear system are satisfied, the presence of a self-loop (either in the existing or controlled interconnections) directly satisfies the coprimeness condition because any path traveling through the node with a self-loop can simply be extended by a single edge (and since $\gcd(k, k + 1) = 1$). For a controlled self-loop, a path of unit length exists, which automatically satisfies the coprimeness condition.

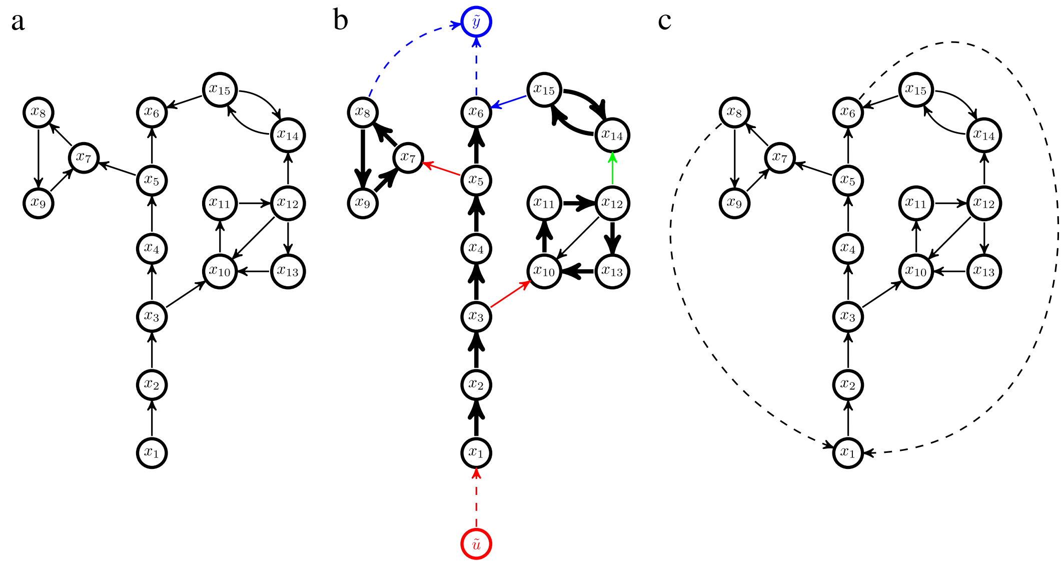

Fig. 4. A 15 node network showing (a) the original graph (b) the resulting control augmented graph of the associated linear system where the bold edges show the matching components, red and blue edges represent distinguished edges for the cacti of $G(\mathbf{A})$ and $G(\mathbf{A}^{\mathrm{T}})$ respectively. The green edge is the distinguished edge common to both of these cacti. (c) the resulting control augmented graph $G(\mathbf{A}, \mathbf{B})$ of the original system. Dashed arrows indicate controlled edges added to the graph. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Remark 16 provides a convenient shortcut to the control configuration algorithm if self-loops exist in the network. In order to guarantee controllability of the bilinear system in Case I without checking the coprimeness condition, only one additional controlled edge is required. It may be worthwhile, then, to compare the computation time of checking the coprime path condition to the cost of inserting another controlled edge in the network.

4.1. Example

Consider the 15 node network as shown in Fig. 4. The maximum matching on $G(\mathbf{A})$ yields a stem $\mathcal{S}_{\mathbf{A}}$ comprised of the nodes $(x_1, x_2, x_3, x_4, x_5, x_6)$; two buds directly attached from the stem namely, $\mathcal{C}_1 = (x_7, x_8, x_9)$ and $\mathcal{C}_2 = (x_{10}, x_{11}, x_{12}, x_{13})$; and the bud $\mathcal{C}_3 = (x_{14}, x_{15})$ attached from the cycle $\mathcal{C}_2$ via the edge (x_{12}, x_{14}). The spanning forest of $G(\mathcal{M}_{\mathbf{A}})$ has one tree rooted in $\mathcal{S}_{\mathbf{A}}$.

The components for $G(\mathbf{A}^{\mathrm{T}})$ are the same, only with the edge orientation reversed. In terms of visualization in Fig. 4 it is easier to just view the graph "from the top" rather than to reverse the edges. While the edge (x_3, x_{10}) connects from the stem $\mathcal{S}_{\mathbf{A}}$ to the bud $\mathcal{C}_2$ (and in turn $\mathcal{C}_3$), a different distinguished edge (x_{15}, x_6) connects from $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$ to $\mathcal{C}_3$ (and in turn $\mathcal{C}_2$). Because $\mathcal{C}_1 \in \mathcal{R}_{\mathbf{A}^{\mathrm{T}}}$ has no distinguished edge from $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$, the spanning forest of $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$ has two trees.

This network is an example of Case III. Therefore, to control the linear system associated with $G(\mathbf{A})$, an edge must be added from the pseudo-input $\tilde{u}$ to x_1. The edge $(\tilde{u}, x_1)$ connects $\tilde{u}$ to all the nodes since there is a single tree in $G(\mathcal{M}_{\mathbf{A}})$. To observe the linear system it is necessary to obtain observations from the roots of both of the trees in the spanning forest of $G(\mathcal{M}_{\mathbf{A}^{\mathrm{T}}})$. We connect to $\tilde{y}$ from the top of the stem $\mathcal{S}_{\mathbf{A}}$ (from node x_6) and from one of the nodes in $\mathcal{C}_1$. Note that the top of $\mathcal{S}_{\mathbf{A}}$ is the bottom of $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$. There will be two edges to $\tilde{y}$ and one from $\tilde{u}$, yielding two controlled edges.

There is ambiguity in which node in $\mathcal{C}_1$ is connected to $\tilde{y}$, however, we can use the coprimeness requirement to help make this choice. In this case the end vertex of the distinguished edge is $x_p = x_7$; the length of the path from $\tilde{u}$ to x_7 is 6. We note that x_7 is not a candidate because the length of the cycle is $c = 3$ and so any path from $\tilde{u}$ to x_7 will always be a multiple of 3. Either other node in $\mathcal{C}_1$ will satisfy the coprimeness condition, therefore, we connect x_8 (i.e., $r = 1$) to $\tilde{y}$, which creates paths of length 7 (no times around $\mathcal{C}_1$) and 10 (one time around $\mathcal{C}_1$). Thus the bilinear network becomes controllable with just two controlled edges, namely (x_6, x_1) and (x_8, x_1).

Although structural controllability analyzes systems based on structure alone, we now demonstrate the general utility of structural control configuration design by synthesizing a control signal to drive the system from an initial state to a target state. In order to do so, we generate a realization of the system with numerical edge weight values. For this scenario we select parameters arbitrarily, setting all edges to be unit value except for $\mathbf{A}_{7,9} = -1$, $\mathbf{A}_{5,4} = \mathbf{A}_{10,13} = \mathbf{A}_{15,14} = 2$, $\mathbf{A}_{13,12} = 3$; we take $T = 15$ to be the total number of steps; and set initial and target states as $\mathbf{x}_0 = [1, \dots, 15]$ and $\mathbf{x}_f = [15, \dots, 1]$. We formulate an optimal control problem that seeks to minimize the sum of squared input values, $J = \sum_{t=0}^{T-1} u^2(t)$ over a time horizon T subject to initial, $\mathbf{x}(0) = \mathbf{x}_0$, and final, $\mathbf{x}(T) = \mathbf{x}_f$, endpoint constraints. Since the bilinear system is a discrete-time system, the optimal control problem is a constrained nonlinear programming problem and can be solved with a variety of open-source and commercial solvers. Fig. 5 presents the optimal control input – the time-varying edge weight for edges (x_6, x_1) and (x_8, x_1) – that drives the system between the desired states with minimum power. If all edge weights in $G(\mathbf{A})$ are instead set to unit value, this realization is uncontrollable because this parameter vector $\boldsymbol{\lambda} = \mathbf{1}$ lies in the Lebesgue measure zero set (proper variety) of the uncontrollable subspace. So although there are specific realizations for which the output of the control configuration design will fail, the set of such realizations lies within a proper variety of the parameter space. In general these pathological cases are more highly nuanced (requiring interdependencies between the values of parameters) than realizations found in physical systems.

5. Conclusions

In this work we have developed both algebraic and graph-theoretic conditions for the structural control of single-input, rank one bilinear systems. In addition, we have used these conditions to develop an algorithm to design the minimal (fewest controlled edges) control configuration for a bilinear network. We find that, in

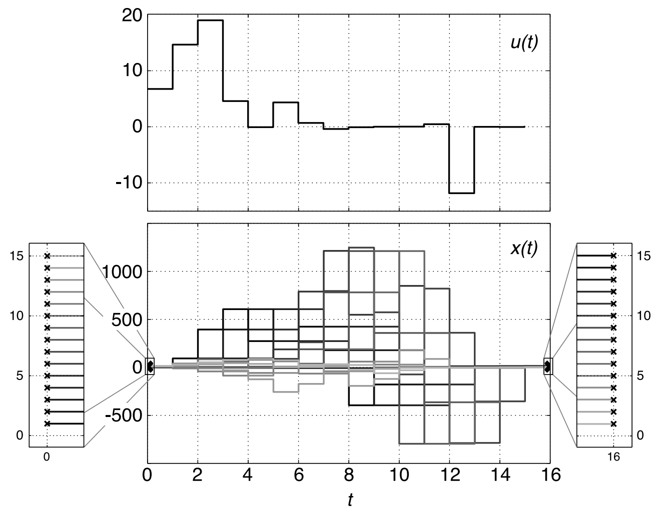

Fig. 5. The 15 node network in Fig. 4 with controlled edges added using our control configuration design algorithm is driven by $u(t)$ from an initial state $\mathbf{x}_0 = [1, \ldots, 15]$ to a final state $\mathbf{x}_1 = [15, \ldots, 1]$ in $T = 15$ steps, starting at $t = 1$. States are colored in different shades of gray to identify that the states have reversed state values.

general, bilinear systems are relatively easy to control in that very few edges need to be added to gain full controllability.

Recent theoretical advances in the control of bilinear systems has been limited to systems with special structure. We believe that the graphical conditions for controllability will help to gain the intuition required to devise conditions for more general structured and non-structured bilinear systems. In particular, we are interested in investigating the controllability (both structured and unstructured) and control configuration design for multi-input bilinear systems. A preliminary class of multi-input systems where all the input matrices are of rank one, is currently under investigation.

Appendix. Proof of technical results

In this section, all the lemmas required for the proof of Theorem 12 and other technical details are presented. We will start by discussing dilations and proving a few interesting results about them. First we introduce the new concept of an atomic dilation for a subset $S \subset \mathcal{X}$.

Definition 17. Let $S \subset \mathcal{X}$ and U be any proper subset of S. Then, the set S is said to possess an atomic dilation if S contains a dilation but U does not contain one. In such a case, we say that $G(\mathbf{A})$ contains an atomic dilation.

An atomic dilation set S is the "smallest set" that contains a dilation. Although it is stated in the literature that a dilation in a graph is equivalent to a rank deficiency in the structured adjacency matrix $\mathbf{A}$, to our knowledge this fact is not formally captured in any paper (Johnston, Barton, & Brisk, 1984). The following lemma formalizes this statement for the case of a single atomic dilation.

Lemma 18. *The generic rank of a $n \times n$ structured matrix $\mathbf{A}$ is $n-1$ if and only if the associated directed graph $G(\mathbf{A})$ has exactly one unique atomic dilation.*

To prove Lemma 18, we need the following which relates atomic dilations to parent sets of nodes.

Lemma 19. *For each atomic dilation set S, we have $|T(S)| = |S| - 1$.*

Proof. Assume S contains an atomic dilation. By definition, then, $|T(S)| < |S|$. Assume $|T(S)| \leq |S| - 2$. Note that $T(S)$ contains all the nodes which have edges incident to any node in S. Now consider a new set $\tilde{S}$ formed by removing a node (any node) from S so that $|\tilde{S}| = |S| - 1$. Let $T(\tilde{S})$ be the corresponding incident set for $\tilde{S}$. Since $\tilde{S} \subset S$, we have $T(\tilde{S}) \subseteq T(S)$, and thus $|T(\tilde{S})| \leq |T(S)| \leq |S| - 2 < |\tilde{S}|$. Thus, $\tilde{S}$ contains a dilation, which is a contradiction with the definition of an atomic dilation. ■

We now present a proof of Lemma 18 that deals with the unicity of atomic dilations and rank deficiency of structured matrices.

Proof of Lemma 18. First, assume that $G(\mathbf{A})$ contains a unique atomic dilation set $S \subseteq \mathcal{X}$. Because a nonzero entry in the jth column of $\mathbf{A}$ indicates that x_j has an incident edge to a node, we can use Lemma 19 to show that $\mathbf{A}|_S$ has at most $|S|-1$ nonzero columns and thus, a generic rank of at most $|S| - 1$. Let us define $\mathbf{A}|_S$ as the matrix comprised of rows of $\mathbf{A}$ restricted to the nodes (states) in S. If S is a singleton, then $\mathbf{A}|_S = \mathbf{0}$ is a zero row, and this immediately proves that the generic rank of $\mathbf{A}|_S$ is $0 = |S|-1$. Assume S contains more than one element and consider a set $\{x_i\} \subset S$. Because S is an atomic dilation set, the proper subset $\{x_i\}$ does not possess a dilation, and so the x_ith row must have at least one nonzero entry. Since x_i was arbitrary, every row in S contains at least one nonzero entry. If $|S| = 2$, then one can easily deduce by this argument that $\mathbf{A}|_S$ has exactly one nonzero column and therefore, possesses a generic rank of one. However, if $|S| > 2$, one can then consider a proper subset $\{x_i, x_j\} \subset S$. Since again this set does not contain a dilation, there will be at least two nonzero columns in $\mathbf{A}|_{\{x_i,x_j\}}$ in addition to the fact that the x_ith and x_jth rows contain at least one nonzero entry each. This implies that the generic rank of $\mathbf{A}|_{\{x_i,x_j\}}$ equals 2. An induction argument then gives us that the generic rank of every subset R of S having $|S|-1$ elements is exactly $|S|-1$. Thus, the generic rank of $\mathbf{A}|_S$ is equal to $|S| - 1$. Using a similar argument, one can then deduce that the generic rank of $\mathbf{A}|_{\mathcal{X}\setminus S}$ equals $|\mathcal{X} \setminus S|$. Also, any proper subset of rows of $\mathbf{A}|_S$ is linearly independent to the rows in $\mathbf{A}|_{\mathcal{X}\setminus S}$ (otherwise that will contradict with the unicity of S). This implies that the generic rank of $\mathbf{A}$ is $n - 1$.

On the other hand, assume that the generic rank of $\mathbf{A}$ is $n - 1$. Identify the smallest set $S \subseteq \mathcal{X}$ such that $\mathbf{A}|_S$ contains linearly dependent rows but the rows in any of the proper subsets of S are linearly independent. If $S = \{x_i\}$ for some $x_i \in \mathcal{X}$, then obviously $\mathbf{A}|_S = \mathbf{0}$ and hence, S is an atomic dilation set. Else, consider a set $\{x_i\} \subset S$. Since $\mathbf{A}|_{\{x_i\}}$ is linearly independent, the x_ith row contains at least one nonzero entry. Again an induction argument shows that no proper subset of S contains a dilation. In particular every subset $R \subset S$ of $|S|-1$ entries has at least $|S|-1$ nonzero columns. In other words, $|T(R)| \geq |R| \geq |S| - 1$. This coupled with the fact that $\mathbf{A}|_S$ contains a set of linearly dependent rows then gives us that $|T(S)| = |S| - 1$. So, S is an atomic dilation set. One can then use a set-theoretic argument to show that S is unique since if there are two atomic dilation sets S and P, then $S \cup P$ will have at least two dilations and correspondingly the generic rank of $\mathbf{A}$ would be at most $n - 2$ leading to a contradiction. ■

Now, assume that the number of non-zero elements in $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$ be N_A, N_c, and N_h, respectively, with $N = N_A + N_c + N_h$ the total number of non-zero elements. Thus, every collection of nonzero entries in the triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h}^\mathrm{T})$ represents a point in $\mathbb{R}^N$ and consequently, a set of parameters for the system. Therefore, $\mathbb{R}^N$ denotes the parameter space for the system.

The following lemma characterizes the condition for the genericity of the property of structural controllability; i.e., it says that structural controllability of a system is equivalent to the fact that all the uncontrollable sets of parameters form a set of Lebesgue measure zero.

Lemma 20. *The system (2) described by the structured triplet* $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ *is structurally controllable if and only if all uncontrollable triplets that are structurally equivalent to* $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ *lie on a proper variety in* $\mathbb{R}^N$.

Proof. Let $\boldsymbol{\lambda} = [\lambda_1 \cdots \lambda_N]^{\mathrm{T}}$ be a parameter vector associated with $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$. Let $I \subset \{1, \ldots, 2n\}$ be defined as the set such that $j \in I$ whenever $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0$; i.e., $I \triangleq \{j\colon \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0, j = 1, \ldots, 2n\}$. Note that whenever $1 \in I$, $\gcd(I) = 1$ automatically. Define S as the set of all sets of coprime integers between 2 and $2n$; i.e., $S = \{\{i_1, \ldots, i_k\}\colon \gcd\{i_1, \ldots, i_k\} = 1, \{i_1, \ldots, i_k\} \subset \{2, \ldots, 2n\}, k \in \{2, \ldots, 2n\}\}$. Define the following polynomials:

$$\psi_1(\boldsymbol{\lambda}) = \det\left([\mathbf{c} \quad \mathbf{Ac} \quad \cdots \mathbf{A}^{n-1}\mathbf{c}]\right)^2;$$

$$\psi_2(\boldsymbol{\lambda}) = \det\left([\mathbf{h} \quad \mathbf{A}^{\mathrm{T}}\mathbf{h} \quad \cdots \quad (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{h}]\right)^2;$$

$$\psi_3(\boldsymbol{\lambda}) = (\mathbf{h}^{\mathrm{T}}\mathbf{c})^2 + \sum_{\{i_1,\ldots,i_k\}\in S} (\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_1-1}\mathbf{c})^2 \cdots (\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_k-1}\mathbf{c})^2.$$

Assume that there exists a controllable $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$; i.e., there exists a $\boldsymbol{\lambda} \in \mathbb{R}^N$ such that the associated system is controllable. In that case as Lemma 11 states, $(\mathbf{A}, \mathbf{c})$ is controllable which implies $\psi_1(\boldsymbol{\lambda}) \not\equiv 0$; $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is observable which implies $\psi_2(\boldsymbol{\lambda}) \not\equiv 0$; and $\gcd(I) = 1$. The last condition implies that there exists a set of coprime integers I between 1 and $2n$ such that for every $j \in I$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0$. Thus, either $1 \in I$ or $I \in S$. Therefore, either $(\mathbf{h}^{\mathrm{T}}\mathbf{c})^2 \not\equiv 0$ or there exists an $\{i_1, \ldots, i_k\} \in S$ with $k \in \{2, \ldots, 2n\}$ such that $(\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_1-1}\mathbf{c})^2 \cdots (\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_k-1}\mathbf{c})^2 \not\equiv 0$ which implies that $\psi_3(\boldsymbol{\lambda}) \not\equiv 0$. Therefore, $\psi(\boldsymbol{\lambda}) \triangleq \psi_1(\boldsymbol{\lambda}) \cdot \psi_2(\boldsymbol{\lambda}) \cdot \psi_3(\boldsymbol{\lambda})$ is such that $\psi(\boldsymbol{\lambda}) \not\equiv 0$. In other words, $\psi(\boldsymbol{\lambda}) = 0$ (which defines the set of all uncontrollable parameters) defines a proper variety.

Conversely, assume that $\psi(\boldsymbol{\lambda}) = 0$ defines a proper variety. This implies that there is a $\boldsymbol{\lambda} \in \mathbb{R}^N$ such that $\psi_1(\boldsymbol{\lambda}) \not\equiv 0$ which implies corresponding $(\mathbf{A}, \mathbf{c})$ is controllable; $\psi_2(\boldsymbol{\lambda}) \not\equiv 0$ which implies that $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is observable; and $\psi_3(\boldsymbol{\lambda}) \not\equiv 0$ which implies that the coprimeness condition is satisfied. Hence, the resulting system is structurally controllable. ■

Before introducing the next lemma, which deals with the generic rank of a matrix, we need to define the concept of linear dependence of structured vectors.

Definition 21. Let $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ be a set of structured vectors of dimension n (with $k \le n$). The set is said to be generically linearly independent if and only if the generic rank of the $k \times n$ matrix $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_k]$ equals k.

In other words, the set of n-dimensional structured vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is generically linearly dependent if and only if, for almost every set of parameter values, there exist constants $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ not all zero, such that $\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = 0$.

Lemma 22. *The generic rank of* $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{h}^{\mathrm{T}} & * \end{bmatrix}$ *equals* $n + 1$ *if and only if the generic ranks of* $[\mathbf{A}\,\mathbf{c}]$ *and* $[\mathbf{A}^{\mathrm{T}}\,\mathbf{h}]^{\mathrm{T}}$ *both equal* n.

Proof. First assume that the generic rank of $\mathbf{M}$ equals $n + 1$. This implies that all the rows (and columns) of $\mathbf{M}$ are generically linearly independent which in turn implies that the generic ranks of $[\mathbf{A}\,\mathbf{c}]$ and $[\mathbf{A}^{\mathrm{T}}\,\mathbf{h}]^{\mathrm{T}}$ (obtained by removing the last row and column, respectively) both equal n. Conversely assume that both $[\mathbf{A}\,\mathbf{c}]$ and $[\mathbf{A}^{\mathrm{T}}\,\mathbf{h}]^{\mathrm{T}}$ both have generic rank n. Then the generic rank of $\mathbf{A}$ is at least $n-1$. If the generic rank of $\mathbf{A}$ equals n, then the free parameter in the $(n+1, n+1)$ position of $\mathbf{M}$ guarantees that the generic rank of $\mathbf{M}$ equals $n + 1$. On the other hand, if the generic rank of $\mathbf{A}$ equals $n - 1$, then $\mathbf{c}$ must be generically linearly independent to all the columns of $\mathbf{A}$. This in turn implies that $[\mathbf{c}^{\mathrm{T}}\,*]^{\mathrm{T}}$ is generically linearly independent to all the columns of $[\mathbf{A}^{\mathrm{T}}\,\mathbf{h}]^{\mathrm{T}}$. Since the columns of $[\mathbf{A}^{\mathrm{T}}\,\mathbf{h}]^{\mathrm{T}}$ are all generically linearly independent by assumption, the $[\mathbf{c}^{\mathrm{T}}\,*]^{\mathrm{T}}$ column makes the generic rank of $\mathbf{M}$ equal $n + 1$. ■

The following lemma characterizes the generic coprimeness condition on the set I (defined in Theorem 12) in terms of walks from $\mathcal{U}$ to $\mathcal{X}_y$ in the directed graph of the system. It states that $\gcd(I) = 1$ is equivalent to the fact that there are closed walks of coprime lengths in $G(\mathbf{A}, \mathbf{B})$ formed by edges in $\mathcal{E}_{\mathbf{A}}$, each with exactly one controlled edge.

Lemma 23. *With I defined as in Theorem 12, the* $\gcd(I) = 1$ *if and only if either there exists a collection of walks of coprime lengths in* $\mathcal{W}_{\mathcal{X}_y\mathcal{U}}$.

Proof. The fact that $\gcd(I) = 1$ implies that there exist integers $\ell_1, \ldots, \ell_k \in I$ such that $\gcd(\{\ell_1, \ldots, \ell_k\}) = 1$. Therefore, we have that $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{\ell_1-1}\mathbf{c} \not\equiv 0, \ldots, \mathbf{h}^{\mathrm{T}}\mathbf{A}^{\ell_k-1}\mathbf{c} \not\equiv 0$. That is there exist pairs of integers $(i_1, j_1), \ldots, (i_k, j_k)$ such that $\mathbf{h}_{j_1}\mathbf{A}^{\ell_1-1}_{j_1i_1}\mathbf{c}_{i_1} \not\equiv 0, \ldots, \mathbf{h}_{j_k}\mathbf{A}^{\ell_k-1}_{j_ki_k}\mathbf{c}_{i_k} \not\equiv 0$ generically. Consider the fact that $\mathbf{h}_{j_1}\mathbf{A}^{\ell_1-1}_{j_1i_1}\mathbf{c}_{i_1} \not\equiv 0$. Since $\mathbf{A}^{\ell_1-1}_{j_1i_1}$ is a scalar, this is equivalent to the fact that $\mathbf{c}_{i_1}\mathbf{h}_{j_1} \not\equiv 0$, so $(\tilde{u}, x_{i_1}) \in \mathcal{E}_{\mathbf{c}}$ and $(x_{j_1}, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}$, and that there exists a sequence of nodes $p_0, \ldots, p_{\ell_1-1} \in \mathcal{X}$ which form a walk W_1 defined as $\tilde{u} \in \mathcal{U} \to x_{i_1} = p_0 \to p_1 \to \cdots \to p_{\ell_1-1} = x_{j_1} \in \mathcal{X}_y$ of length ℓ_1. Therefore, $W_1 \in \mathcal{W}_{\mathcal{X}_y\mathcal{U}}$. Also, $\ell_1 \in I$ implies that $\ell_1 \le 2n$. Using a similar procedure we can construct walks $W_2, \ldots, W_k \in \mathcal{W}_{\mathcal{X}_y\mathcal{U}}$ of lengths $\ell_2, \ldots, \ell_k$, respectively. Since $\gcd(\{\ell_1, \ldots, \ell_k\}) = 1$, the walks are coprime. ■

The final result of this section shows that if $(\mathbf{A}, \mathbf{c})$ is structurally controllable or $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is structurally observable, the set I is not empty. In fact there exists an integer $k \le n$ such that $k \in I$; i.e., there exists at least one walk of length less than or equal to n satisfying the conditions in the previous lemma. This is due to the fact if $(\mathbf{A}, \mathbf{c})$ is structurally controllable (if $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is structurally observable) then $\tilde{u}$ is connected to ($\tilde{y}$ is connected from) every vertex in $\mathcal{X}$, thus there exists at least one walk from $\tilde{u}$ to $\tilde{y}$.

Lemma 24. *Under the condition that* $(\mathbf{A}, \mathbf{c})$ *is structurally controllable or* $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ *is structurally observable, there exists a* $k \le n$ *such that* $k \in I$.

Proof. First assume that $(\mathbf{A}, \mathbf{c})$ is structurally controllable. Then the generic rank of $[\mathbf{c}\ \mathbf{Ac}\ \cdots\ \mathbf{A}^{n-1}\mathbf{c}]$ equals n. This is equivalent to saying that for any structured row vector $\mathbf{h}^{\mathrm{T}}$ such that $\mathbf{h} \not\equiv \mathbf{0}$, $\mathbf{h}^{\mathrm{T}}[\mathbf{c}\ \mathbf{Ac}\ \cdots\ \mathbf{A}^{n-1}\mathbf{c}]$ is not zero identically. In other words, there exists at least one $k \in \{1, \ldots, n\}$ such that $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{k-1}\mathbf{c} \not\equiv 0$, and hence $k \in I$. The proof for the situation when $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is observable is almost identical. ■

This result also implies the fact that under the assumption of controllability and/or observability of the associated linear system, the set $\mathcal{W}_{\mathcal{X}_y\mathcal{U}}$ defined using the graph of the bilinear system is nonempty.

A.1. Proof of Theorem 12

Proof of 1 ⇔ 2**.** The structural controllability of the structured triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h}^{\mathrm{T}})$ is equivalent to saying that the generic ranks of $[\mathbf{c}\ \mathbf{Ac}\ \cdots\ \mathbf{A}^{n-1}\mathbf{c}]$ and $[\mathbf{h}\ \mathbf{A}^{\mathrm{T}}\mathbf{h}\ \cdots\ (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{h}]$ both equal n, and $\gcd(I) = 1$. The generic rank conditions are equivalent to saying that $(\mathbf{A}, \mathbf{c})$ is structurally controllable and $(\mathbf{A}, \mathbf{h}^{\mathrm{T}})$ is structurally observable. Using the results from Dion et al. (2003), Shields and Pearson (1976), alternatively one can say that the generic ranks of $[\mathbf{A}\,\mathbf{c}]$ and $[\mathbf{A}^{\mathrm{T}}\,\mathbf{h}]^{\mathrm{T}}$ equal n and that $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are irreducible. Using Lemma 22, this is equivalent to the fact that the generic rank of $\mathbf{M}$ equals $n + 1$ and that the irreducibility and coprimeness conditions hold true.

Proof of 2 ⇔ 3**.** The first part of the equivalence follows from Lemma 22 and the results in Dion et al. (2003), Reinschke (1988). The second part of the equivalence follows from Lemma 23. ■

We finish off Appendix by presenting a sketch of the proof of Proposition 15 which deals with the special case of strongly connected graphs.

Proof of Proposition 15. Since $\mathbf{A}$ is of full rank, the generic rank conditions stated in 2a of Theorem 12 are automatically satisfied. Also, the strong connectedness of $\mathbf{A}$ automatically satisfies the irreducibility conditions for $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$. This also guarantees that node x_i (with the self-loop; i.e., $\mathbf{A}_{ii} \not\equiv 0$) lies on one of walks from $\tilde{u}$ to $\tilde{y}$. One can easily construct two walks by ignoring the self-loop and by counting the self-loop once which differ in length by one and, therefore, are coprime to each other. Thus, the structured bilinear system is structurally controllable. ■

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Supratim Ghosh is a postdoctoral research fellow in the Engineering Systems and Design pillar at Singapore University of Technology and Design. He received his undergraduate degree in Instrumentation and Control Engineering from National Institute of Technology, Jalandhar, India. He then went on to receive his M.S. in Electrical Engineering, M.A. in Mathematics and Ph.D. in Electrical Engineering all from the Pennsylvania State University, USA. His current research interests include decentralized and networked control, game theory, multi-agent systems, and their applications to problems in networks, power systems, and bio-medicine.

Justin Ruths is an assistant professor in the Engineering Systems Design pillar at the Singapore University of Technology and Design. He received a B.S. from Rice University, Houston, TX, USA; M.S. from Columbia University, New York, NY, USA; and Ph.D. degree in Systems Science and Applied Mathematics from Washington University, St. Louis, MO, USA. His research focuses on the control and optimization of large-scale connected systems, including problems in neuroscience, quantum control, cyber-physical system security, and control of networks.