On Structural Controllability of a Class of Bilinear Systems

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***Abstract*— We provide the necessary and sufficient conditions for structural controllability of discrete-time single-input bilinear systems with an input matrix of rank one. These come in the form of two equivalent conditions: one involves checking the generic rank and irreducibility of structured system matrices, and the other involves checking for walks within the corresponding directed graph representation of the system connectivity.**

I. Introduction

The topic of structural control has resurfaced due to recent developments in the control of complex networks - see [1] and the references therein. Structural control, which provides tools to analyze systems on the level of connectivity (i.e., parameter-independent), offers two key advantages over classical results in control theory. Due to the sheer size of real-world networks, it is difficult (or impossible) to accurately survey the parameters that govern their dynamics. Moreover, the values of these parameters are likely to change over time because many of these networks are influenced by their environments, e.g., biochemical, ecological, organizational, and social networks are typically not as static as engineered systems. Secondly, again due to the size of these systems, structural methods provide an attractive alternative to the more computationally costly conventional control methods.

Work on structured systems has almost exclusively focused on the topic of linear systems [2]–[6]. Because of this, research on the control of complex networks has been limited to considering linear systems with exogenous inputs [1], [7]. However, in many cases it is both more appropriate and more feasible to consider controlling the edges of the network rather than imposing an external control. For example, in biochemical networks, medical drugs offer the ability to block the interactions between proteins (i.e., modulate the edges of network), but directly adjusting the concentration of specific proteins (i.e., directly controlling a node of the network) is a much more difficult and invasive procedure. The notion of controlling edges in a network is best captured by a model with bilinear dynamics.

Bilinear systems are a class of nonlinear systems such that the system dynamics are linear in state for fixed external inputs and linear in inputs for fixed states. Bilinear models have been used to describe a variety of real-world systems such as switched circuits, population growth models, social networks, power transmission lines, and quantum spin dynamics [8]–[11]. While the observability of bilinear systems [10], [12] has been characterized completely, an exact characterization of controllability of such systems is still an open question [9]. In contrast to linear systems, the controllability of a bilinear system is not a property dual to observability and, therefore, requires deeper analysis. Nevertheless, several sufficient conditions for controllability of bilinear systems are available [13]–[15]. The most complete characterization of controllability with algebraic conditions is known for discrete-time bilinear systems with a single input and where the input matrix is of rank one [16]–[18]. Conditions are known for a class of unconstrained bilinear systems without a rank restriction, however, these rely on Lie Algebra [9].

For structured bilinear systems, similar to above, only observability analysis has received attention [19], [20]. Controllability analysis of structured bilinear systems is absent from the literature.

Leveraging on the algebraic conditions that exist for unstructured bilinear systems, in this paper, we present two equivalent, easily verifiable characterizations of controllability of structured discrete-time homogeneous bilinear systems with rank-one input matrices. The first characterization is algebraic in nature and involves checking the generic rank and irreducibility of system matrices similar to the ones required for linear systems with some additional conditions. In addition, we develop a set of equivalent graph-theoretic characterizations of controllability that involves checking for walks in a directed graph. Analogous to the presentation in [16], a structured discrete-time rank-one bilinear system can be decomposed into a structured linear system with a feedback compensator. The controllability and observability of the associated linear system is necessary for the the controllability of the overall bilinear system.

Notation. The (i,j)th entry of $\mathbf{A}^k$ is denoted as a_{ij}^k. A pair $(\mathbf{A},\mathbf{c})$ is called controllable if the associated controllability matrix $\begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \cdots & \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix}$ has full row rank. Similarly, a pair $(\mathbf{A},\mathbf{h})$ is called observable if the associated observability matrix $\begin{bmatrix}\mathbf{h} & \mathbf{A}^{\mathrm{T}}\mathbf{h} & \cdots & (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{h}\end{bmatrix}^{\mathrm{T}}$ has full row rank. Similarly a triplet $(\mathbf{A},\mathbf{c},\mathbf{h})$ is said to be controllable and observable if both the rank conditions are satisfied.

II. A Graph Theoretic Model of Structured Bilinear Systems

We are concerned with bilinear systems whose state evolution is described by the following

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + u(t)\mathbf{B}\mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}$ denotes the external control input and the matrices $\mathbf{A},\mathbf{B} \in \mathbb{R}^{n\times n}$ denote the state matrix and input matrix, respectively.

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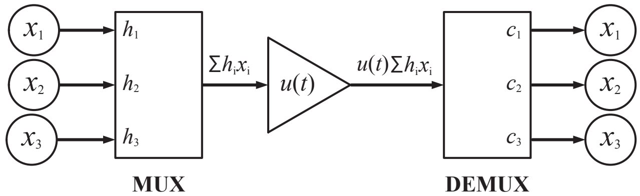

Fig. 1. An example of a physical realization of a rank-one homogeneous bilinear system.

Similar to the systems described in [16], [17], we focus on systems where $\mathbf{B}$ is of rank one, such that the input matrix can be written as $\mathbf{B} = \mathbf{c}\mathbf{h}^{\mathrm{T}}$ where $\mathbf{c}, \mathbf{h} \in \mathbb{R}^n$. Thus, we have an alternate representation of the system described by (1)

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + u(t)\mathbf{c}\mathbf{h}^{\mathrm{T}}\mathbf{x}(t). \tag{2}$$

Although the rank one assumption placed on the model is restrictive, it still applies to non-trivial class of applications. One of the more complex examples of a real system with a rank one input matrix, i.e., of the form in (2), is shown in Fig. 1 and is comprised of a multiplexer, an amplifier (the gain of which is the external control input), and a de-multiplexer. This kind of architecture appears in long-haul transmission systems where the bandwidth is limited to and from a remotely located controller. The multiplexer receives the outputs (or measurements) from a set of nodes (the nonzero elements of $\mathbf{h}$) and combines them into a single stream to be sent to the amplifier, $\sum_{i=1}^{n} h_i x_i(t) = \mathbf{h}^{\mathrm{T}}\mathbf{x}(t)$. This data stream is then amplified using the external control input, $u(t)$, generated by the designer and then sent to a set of nodes (the nonzero elements of $\mathbf{c}$) via the de-multiplexer.

A. Structured Systems

The concept of structured systems allows us to model systems when the parameters that define their dynamics are unknown. In other words, for the matrices $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$ in (2), only the locations of zero and non-zero (denoted by $*$) entries are known, but not the exact values of non-zero entries. In modeling such a system, the locations of fixed zeros are conserved, but the non-zero entries are replaced by free independent parameters. A simple example of such matrices could look like

$$\mathbf{A} = \begin{bmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ \lambda_3 \end{bmatrix} \quad \mathbf{h}^{\mathrm{T}} = \begin{bmatrix} \lambda_4 & \lambda_5 \end{bmatrix},$$

where $\lambda_i \in \mathbb{R}$, $i \in \{1, \ldots, 5\}$, represent the structural connectivity (i.e., sparsity pattern) of a system described in (2). We study these systems in a generic sense, such that their properties hold for almost all potential choices of the independent parameters.

Definition 1: A property (e.g., controllability, stability) is said to be generic if the set of values of parameters for which the property does not hold is a set of Lebesgue measure zero.

One of the most straightforward techniques to show that a set is of Lebesgue measure zero is to show that this set is the zero set of one or more polynomials with real coefficients, i.e., a proper algebraic variety.

Due to the relaxation delivered by considering the system generically, the conditions for structural properties tend to be simpler than their classical analogs. This simplification makes studying the control related properties of large scale systems, like complex networks, tractable.

Definition 2: A structured system (2) described by $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is said to be structurally equivalent to another system $(\bar{\mathbf{A}}, \bar{\mathbf{c}}, \bar{\mathbf{h}})$ if there is a one-to-one correspondence between the locations of fixed zero and non-zero entries of $\mathbf{A}$ and $\bar{\mathbf{A}}$, $\mathbf{c}$ and $\bar{\mathbf{c}}$, and $\mathbf{h}$ and $\bar{\mathbf{h}}$, respectively.

Definition 3: The system (2) described by the triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is said to be structurally controllable if there exists a triplet $(\bar{\mathbf{A}}, \bar{\mathbf{c}}, \bar{\mathbf{h}})$ which is structurally equivalent to $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ and is controllable.

B. Graph Representation of Structured Bilinear Systems

Directed graphs offer a natural alternate representation of structured systems. In this subsection, we introduce a graph-theoretic representation of structured bilinear systems derived from the corresponding description of structured linear systems [2], [4], [6]. In order to do so, we first define a linear system *associated* to the bilinear system (2) (similar to that in [16]),

$$\begin{aligned} \tilde{\mathbf{x}}(t+1) &= \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{c}\tilde{u}(t) \\ \tilde{y}(t) &= \mathbf{h}^{\mathrm{T}}\tilde{\mathbf{x}}(t). \end{aligned} \tag{3}$$

Here, $\tilde{\mathbf{x}}(t)$, $\tilde{u}(t)$, and $\tilde{y}(t)$ define the pseudo-state, pseudo-input and the pseudo-observation of the system, respectively. The variables $\tilde{u}(t)$ and $\tilde{y}(t)$ play an important role in the graph-theoretic representation of (2).

The directed graph $G = (\mathcal{V}, \mathcal{E})$ describing (3) comprises the vertices ($\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}$) corresponding to the states of the system $\mathcal{X} = \{x_1, \ldots, x_n\}$, pseudo-input $\mathcal{U} = \{\tilde{u}\}$, and the pseudo-observation $\mathcal{Y} = \{\tilde{y}\}$; and edges ($\mathcal{E} = \mathcal{E}_{\mathbf{A}} \cup \mathcal{E}_{\mathbf{c}} \cup \mathcal{E}_{\mathbf{h}}$) corresponding to the matrices $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$, where $\mathcal{E}_{\mathbf{A}} = \{(x_i, x_j) \colon a_{ji} \not\equiv 0\}$, $\mathcal{E}_{\mathbf{c}} = \{(\tilde{u}, x_j) \colon c_j \not\equiv 0\}$, and $\mathcal{E}_{\mathbf{h}} = \{(x_i, \tilde{y}) \colon h_i \not\equiv 0\}$, respectively. Here we use the operator $\not\equiv$ to express that the operation holds generically.

Whenever there exists a pair $(i, j) \in \mathcal{X}^2$ in system (2) such that $c_j h_i \not\equiv 0$ ($\mathbf{B}_{ji} \not\equiv 0$), there exist edges $(x_i, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}$ and $(\tilde{u}, x_j) \in \mathcal{E}_{\mathbf{c}}$ in system (3). In this case we say that there exists a *controlled edge* from x_i to x_j. This represents an interconnection between the nodes whose strength (edge weight) can be externally controlled using the input $u(t)$. If there exists an $x_i \in \mathcal{X}$ such that $c_i h_i \not\equiv 0$ ($\mathbf{B}_{ii} \not\equiv 0$), we say that the network possesses a *controlled self-loop*.

A *walk* W in a graph is a sequence of edges such that the end vertex of a preceding edge is the begin vertex of the next. The length of a walk is the number of edges present in the walk. A walk is *closed* if its begin and end vertex are the same. An ℓ-length walk $W = \{(w_0, w_1), \ldots, (w_{\ell-1}, w_\ell)\}$ may also be represented as $w_0 \to w_1 \to \cdots w_{\ell-1} \to w_\ell$. A *path* is a walk when none of the vertices and edges are repeated. Two paths (walks) are called disjoint if they consist of disjoint sets of vertices. A path is called a $\mathcal{U}$-rooted ($\mathcal{Y}$-topped) path if the path has its begin (end) vertex in $\mathcal{U}$ ($\mathcal{Y}$). A number of mutually disjoint $\mathcal{U}$-rooted ($\mathcal{Y}$-topped) paths is called a $\mathcal{U}$-rooted ($\mathcal{Y}$-topped) path family. A closed path is called a cycle. A set of disjoint cycles is called a cycle family. We say a collection of walks (or paths or cycles) $\mathcal{W}$ *covers* all the vertices in $\mathcal{X}$ if every vertex in $\mathcal{X}$ is either the start or the end vertex of at least one of the edges in $\mathcal{W}$.

Define the set $\mathcal{X}_\mathcal{Y}$ as the set of state vertices which have a directed edge to $\tilde{y}$; i.e., $\mathcal{X}_\mathcal{Y} = \{x_j \in \mathcal{X} \colon h_j \not\equiv 0\}$. Let $\mathcal{W}_{\mathcal{X}_\mathcal{Y}\mathcal{U}}$ denote the set of all walks from $\tilde{u}$ to any vertex $x_j \in \mathcal{X}_\mathcal{Y}$ of length less than or equal to n^2. Note that the vertices and/or edges in such a walk may be repeated. As we will see later, under the assumption that $(\mathbf{A}, \mathbf{c})$ is structurally controllable and/or $(\mathbf{A}, \mathbf{h})$ is observable, the set $\mathcal{W}_{\mathcal{X}_\mathcal{Y}\mathcal{U}}$ is always nonempty.

C. Structural Controllability of Linear Systems

Our results on structural controllability of bilinear systems, draws on several existing results in structural controllability of linear systems [2]–[4], therefore, we review the key terminology and results here. A structured linear system is described by

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times n}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are structured matrices.

Definition 4: The generic rank of a structured matrix $\mathbf{M}$ is the maximal possible rank obtained by fixing the parameters of $\mathbf{M}$.

Definition 5: The system (4) defined by structured matrices $(\mathbf{A}, \mathbf{B})$ is said to be reducible if and only if there exists a permutation matrix $\mathbf{P}$ such that the pair $(\mathbf{A}, \mathbf{B})$ satisfies

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \text{and} \quad \mathbf{P}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}.$$

The following result establishes the criteria for structural controllability of linear systems, connecting the algebraic and graphical representations [2]–[6].

Lemma 1: For the linear system (4) described by structured matrices $(\mathbf{A}, \mathbf{B})$ the following are equivalent:

1. The system (4) is structurally controllable.
2. The generic rank of $\begin{bmatrix}\mathbf{A} & \mathbf{B}\end{bmatrix}$ equals n and the pair $(\mathbf{A}, \mathbf{B})$ is irreducible.
3. In G there exists a $\mathcal{U}$-rooted path to every vertex $x_i \in \mathcal{X}$ and there exists a disjoint union of a $\mathcal{U}$-rooted path family and a cycle family that covers all the state vertices in $\mathcal{X}$.

Similar results exist for the observability of the structured matrices $(\mathbf{A}, \mathbf{C})$.

D. Controllability of Bilinear Systems

Our definition of generic properties says that the property must hold for almost all choices of parameter values. However, in order to verify this, the classic (unstructured) analog of this property must exist and be well-defined. Such verifiable, algebraic conditions, in the most general sense, for controllability bilinear systems exist only for the class of single input, rank one described systems by (2) [16], [17]. Our main result builds from the following result.

Lemma 2: The bilinear system (2) described by $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is controllable if and only if both the following hold:

1. $\operatorname{rank}\begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \cdots & \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{h}^{\mathrm{T}} \\ \mathbf{h}^{\mathrm{T}}\mathbf{A} \\ \cdots \\ \mathbf{h}^{\mathrm{T}}\mathbf{A}^{n-1}\end{bmatrix} = n,$
2. The greatest common divisor (g.c.d.) of all the elements in I equals one where $I = \{j \colon \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \neq 0,\ j = 1, \ldots, n^2\}$.

III. Main Result

The following result charaterizes the controllability of structured bilinear systems (2) described by the matrix triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ such that only the sparsity structure of the matrices is known.

Theorem 1: For the system described by (2) and the structured triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ the following are equivalent:

1. The system (2) is structurally controllable.

2a. The generic rank of $\mathbf{M} = \begin{bmatrix}\mathbf{A} & \mathbf{c} \\ \mathbf{h}^{\mathrm{T}} & * \end{bmatrix}$ equals $n+1$; $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are irreducible.

b. We have $\gcd(I) = 1$ where $I = \{j \colon \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0 \text{ with } j = 1, \ldots, n^2\}$.

3a. In G the collection of all the paths from $\mathcal{U}$ to $\mathcal{Y}$ cover all the vertices in $\mathcal{X}$ and there exists a disjoint union of a $\mathcal{U}$-rooted and $\mathcal{Y}$-topped path family and cycle family that covers all the state vertices.

b. There exist a collection of walks of coprime lengths in $\mathcal{W}_{\mathcal{X}_\mathcal{Y}\mathcal{U}}$.

Remark. The structural controllability and observability of the associated linear system (3) is necessary for structural controllability of the bilinear system and is provided by parts 2a, and equivalently, 3a of the theorem. Part 3b of the theorem presents a graphical equivalence of part 2b which together with 3a (correspondingly 2a) completely characterizes the structural controllability of rank-one bilinear systems. Part 3b indicates that there must exist closed walks of coprime lengths, each having exactly one controlled edge. Note that part 3b is automatically satisfied if there exists a controlled self-loop. The proof for Theorem 1 is provided in Section IV.

A special case of this result exists if the graph represented by $\mathbf{A}$ is strongly connected with n self-loops; i.e., $a_{ii} \not\equiv 0$ for all $i \in \{1, \ldots, n\}$. In this case, the bilinear system described by $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is structurally controllable for any non-zero $\mathbf{c}$ and $\mathbf{h}$ (if $\mathbf{c}$ or $\mathbf{h}$ is zero, then that would imply $\mathbf{B} \equiv \mathbf{0}$ which would make the system uncontrollable for any input). This is formally stated in the proposition below.

Proposition 1: Suppose $(\mathcal{X}, \mathcal{E}_\mathbf{A})$ is strongly connected and $(x_i, x_i) \in \mathcal{E}_\mathbf{A}$ for every $i \in \{1, \ldots, n\}$. Then the bilinear system described by (2) is structurally controllable for any structured $\mathbf{c}$ and $\mathbf{h}$ such that $\mathbf{B} = \mathbf{c}\mathbf{h}^{\mathrm{T}} \not\equiv \mathbf{0}$.

Proof: We will only present a sketch of the proof here. Since a_{ii} is not a fixed zero for every i, the generic rank of $\mathbf{M}$ trivially equals $n+1$. Also, $(\mathcal{X}, \mathcal{E}_{\mathbf{A}})$ being strongly connected implies that that every state vertex lies on one of the directed paths from $\mathcal{U}$ to $\mathcal{Y}$. This, in addition to self-loops, guarantees the irreducibility of $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$. Furthermore, since $\mathbf{B} \not\equiv \mathbf{0}$, there exists at least one pair $(i, j) \in \{1, \ldots, n\}^2$ such that $(x_i, \tilde{y}), (\tilde{u}, x_j) \in \mathcal{E}$; i.e., there is a controlled edge from x_i to x_j. Since $(\mathcal{X}, \mathcal{E}_{\mathbf{A}})$ is strongly connected, there exists a walk from x_j to x_i using only the edges of $\mathcal{E}_{\mathbf{A}}$. Let the length of the walk be ℓ so that the overall walk W_1 starting from the edge $(\tilde{u}, x_j)$ has a length of $\ell+1$. Define another walk W_2 using the same edges with $(x_j, x_j) \in \mathcal{E}_{\mathbf{A}}$ added. In that case W_2 will be of length $\ell+2$, which is comprime to the the length of W_1. ■

Note that if in addition $\mathbf{A}$ is nonsingular, only a single self-loop is required. This proposition automatically applies to connected undirected (bidirectional) graphs, since every connected undirected graph is strongly connected.

IV. Proof of Main Result

We decompose the proof of the main result into showing the equivalence relation between pairs of conditions. We begin by presenting several lemmas which simplify the main proof.

A. Lemmas

Let the number of non-zero elements in $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{h}$ be N_A, N_c, and N_h, respectively, with $N = N_A + N_c + N_h$ the total number of non-zero elements. Thus every point in $\mathbb{R}^N$ represents a set of parameters for the system, and consequently $\mathbb{R}^N$ denotes the parameter space for the system.

An algebraic variety is the zero set of a finite set of polynomials. An algebraic variety $V \subset \mathbb{R}^N$ is called a proper variety if $V \neq \mathbb{R}^N$ and nontrival if $V \neq \emptyset$. A property on $\mathbb{R}^N$ is called a *generic* property if the set of values in $\mathbb{R}^N$ not satisfying the property lie on a proper variety [3].

Lemma 3: The system (2) described by the structured triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is structurally controllable if and only if all uncontrollable triplets structurally equivalent to $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ lie on a proper variety in $\mathbb{R}^N$.

Proof: Let $\lambda = \begin{bmatrix}\lambda_1 & \cdots & \lambda_N\end{bmatrix}^{\mathrm{T}}$ be a parameter vector associated with $(\mathbf{A}, \mathbf{c}, \mathbf{h})$. Note that whenever $1 \in I$, $\gcd(I) = 1$ automatically. Define S as the set of all sets of coprime integers between 2 and n^2; i.e., $S = \{\{i_1, \ldots, i_k\} \colon \gcd\{i_1, \ldots, i_k\} = 1, \{i_1, \ldots, i_k\} \subset \{2, \ldots, n^2\}, k \in \{2, \ldots, n^2\}\}$. Define the following polynomials:

$$\psi_1(\lambda) = \det\left(\begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \cdots \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix}\right)^2;$$

$$\psi_2(\lambda) = \det\left(\begin{bmatrix}\mathbf{h} & \mathbf{A}^{\mathrm{T}}\mathbf{h} & \cdots & \left(\mathbf{A}^{\mathrm{T}}\right)^{n-1}\mathbf{h}\end{bmatrix}\right)^2;$$

$$\psi_3(\lambda) = (\mathbf{h}^{\mathrm{T}}\mathbf{c})^2 + \sum_{\{i_1,\ldots,i_k\}\in S} (\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_1-1}\mathbf{c})^2 \cdots (\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_k-1}\mathbf{c})^2.$$

Assume that there exists a controllable triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$; i.e., there exists a $\lambda \in \mathbb{R}^N$ such that the associated system is controllable. In that case as Lemma 2 states, $(\mathbf{A}, \mathbf{c})$ is controllable which implies $\psi_1(\lambda) \not\equiv 0$; $(\mathbf{A}, \mathbf{h})$ is observable which implies $\psi_2(\lambda) \not\equiv 0$; and $\gcd(I) = 1$ where $I = \{j \colon \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0; j = 1, \ldots, n^2\}$. The last condition implies that there exists a set of coprime integers I between 1 and n^2 such that for every $j \in I$, $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0$. By definition, either $1 \in I$ or $I \in S$. Therefore, either $(\mathbf{h}^{\mathrm{T}}\mathbf{c})^2 \not\equiv 0$ or there exists an $\{i_1, \ldots, i_k\} \in S$ with $k \in \{2, \ldots, n^2\}$ such that $(\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_1-1}\mathbf{c})^2 \cdots (\mathbf{h}^{\mathrm{T}}\mathbf{A}^{i_k-1}\mathbf{c})^2 \not\equiv 0$ which implies that $\psi_3(\lambda) \not\equiv 0$. Therefore, $\psi(\lambda) \triangleq \psi_1(\lambda) \cdot \psi_2(\lambda) \cdot \psi_3(\lambda)$ is such that $\psi(\lambda) \not\equiv 0$. In other words, $\psi(\lambda) = 0$ (which defines the set of all uncontrollable parameters) defines a proper variety.

Conversely, assume that $\psi(\lambda) = 0$ defines a proper variety. This implies that there is a $\lambda \in \mathbb{R}^N$ such that $\psi_1(\lambda) \not\equiv 0$ which implies corresponding $(\mathbf{A}, \mathbf{c})$ is controllable; $\psi_2(\lambda) \not\equiv 0$ which implies that $(\mathbf{A}, \mathbf{h})$ is observable; and $\psi_3(\lambda) \not\equiv 0$ which implies that the g.c.d. condition is satisfied. Hence, the resulting system is structurally controllable. ■

Before introducing the next lemma, which deals with the generic rank of a matrix, we need to define the concept of linear dependence of structured vectors.

Definition 6: Let $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ be a set of structured vectors of dimension n (with $k \leq n$). The set is said to be generically linearly independent if and only if the generic rank of the $k \times n$ matrix $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_k\end{bmatrix}$ equals k.

In other words, the set of n-dimensional structured vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is generically linearly dependent if and only if there for almost every set of parameter values, there exist constants $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ not all zero, such that $\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = 0$.

Lemma 4: The generic rank of $\mathbf{M} = \begin{bmatrix}\mathbf{A} & \mathbf{c} \\ \mathbf{h}^{\mathrm{T}} & *\end{bmatrix}$ equals $n+1$ if and only if the generic ranks of $\begin{bmatrix}\mathbf{A} & \mathbf{c}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{A} \\ \mathbf{h}^{\mathrm{T}}\end{bmatrix}$ both equal n.

Proof: First assume that the generic rank of $\mathbf{M}$ equals $n+1$. This implies that all the rows (and columns) of $\mathbf{M}$ are generically linearly independent which in turn implies that the generic ranks of $\begin{bmatrix}\mathbf{A} & \mathbf{c}\end{bmatrix}$ (obtained by removing the last row) and $\begin{bmatrix}\mathbf{A}^{\mathrm{T}} & \mathbf{h}\end{bmatrix}^{\mathrm{T}}$ (obtained by removing the last column) both equal n. Conversely assume that both $\begin{bmatrix}\mathbf{A} & \mathbf{c}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{A}^{\mathrm{T}} & \mathbf{h}\end{bmatrix}^{\mathrm{T}}$ both have generic rank n. Then the generic rank of $\mathbf{A}$ is at least $n-1$. If the generic rank of $\mathbf{A}$ equals n, then the free parameter in the free parameter in the $(n+1, n+1)$ position of $\mathbf{M}$ guarantees that the generic rank of $\mathbf{M}$ equals $n+1$. On the other hand, if the generic rank of $\mathbf{A}$ equals $n-1$, then $\mathbf{c}$ is generically linearly independent to all the columns of $\mathbf{A}$. This in turn implies that $\begin{bmatrix}\mathbf{c}^{\mathrm{T}} & *\end{bmatrix}^{\mathrm{T}}$ is generically linearly independent to all the columns of $\begin{bmatrix}\mathbf{A}^{\mathrm{T}} & \mathbf{h}\end{bmatrix}^{\mathrm{T}}$. But the columns of $\begin{bmatrix}\mathbf{A}^{\mathrm{T}} & \mathbf{h}\end{bmatrix}^{\mathrm{T}}$ are all generically linearly independent by assumption. Thus, the generic rank of $\mathbf{M}$ equals $n+1$. ■

The following lemma characterizes the generic g.c.d condition on the set I (defined in Theorem 1) in terms of walks from $\mathcal{U}$ to $\mathcal{X}_{\mathcal{Y}}$ in the equivalent directed graph of the system. It states that $\gcd(I) = 1$ is equivalent to the case there are closed walks of coprime lengths in G formed by edges in

$\mathcal{E}_{\mathbf{A}}$, each with exactly one controlled edge.

Lemma 5: With I defined as in Theorem 1, the $\gcd(I) = 1$ if and only if either there exist walks of coprime lengths in $\mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$.

Proof: The fact that $\gcd(I) = 1$ implies that there exist integers $\ell_1, \ldots, \ell_k \in I$ such that $\gcd(\{\ell_1, \ldots, \ell_k\}) = 1$. Therefore, we have that $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{\ell_1-1}\mathbf{c} \not\equiv 0, \ldots, \mathbf{h}^{\mathrm{T}}\mathbf{A}^{\ell_k-1}\mathbf{c} \not\equiv 0$ generically. That is there exist pairs of integers $(i_1, j_1), \ldots, (i_k, j_k)$ such that $h_{j_1} a_{j_1 i_1}^{\ell_1-1} c_{i_1} \not\equiv 0, \ldots, h_{j_k} a_{j_k i_k}^{\ell_k-1} c_{i_k} \not\equiv 0$ generically. Consider the fact that $h_{j_1} a_{j_1 i_1}^{\ell_1-1} c_{i_1} \not\equiv 0$. This is in turn equivalent to the fact that $c_{i_1} h_{j_1} \not\equiv 0$, so $(\tilde{u}, x_{i_1}) \in \mathcal{E}_{\mathbf{c}}$ and $(x_{j_1}, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}$, and that there exist a sequence of nodes $p_0, \ldots, p_{\ell_1-1} \in \mathcal{X}$ such that $p_0 = x_{i_1}$, $p_{\ell_1-1} = x_{j_1}$, and $(x_{p_s}, x_{p_{s+1}}) \in \mathcal{E}_{\mathbf{A}}$ for $s \in \{0, \ldots, \ell_1 - 2\}$. In other words there exists a walk W_1 defined as $\tilde{u} \in \mathcal{U} \to x_{i_1} = p_0 \to p_1 \to \cdots \to p_{\ell_1-1} = x_{j_1} \in \mathcal{X}_{\mathcal{Y}}$ of length ℓ_1. Therefore, $W_1 \in \mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$. Also, $\ell_1 \in I$ implies that $\ell_1 \leq n^2$. Using a similar procedure we can construct walks $W_2, \ldots, W_k \in \mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$ of lengths $\ell_2, \ldots, \ell_k$, respectively. Since $\gcd(\{\ell_1, \ldots, \ell_k\}) = 1$, the proof is complete. ■

As stated eariler, whenever $1 \in I$; i.e., there exists a controlled self-loop or equivalently an $x_j \in \mathcal{V}$ such that there exists a walk $\tilde{u} \to x_j \to \tilde{y}$ in G, Lemma 5 is automatically satisfied. Furthermore, if there exists a self-loop on a node in any one of the walks from $\tilde{u}$ to $\mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$, Lemma 5 is also satisfied because if such a walk exists, we can define a second walk by including the self-loop just once which will be coprime in length to the previous walk.

The final result of this section shows that if $(\mathbf{A}, \mathbf{c})$ is structurally controllable or $(\mathbf{A}, \mathbf{h})$ is structurally observable, the set I is not empty. In fact there exists an integer $k \leq n$ such that $k \in I$; i.e., there exists at least one walk of length less than or equal to n satisfying the conditions in the previous lemma. This is due to the fact if $(\mathbf{A}, \mathbf{c})$ is structurally controllable (if $(\mathbf{A}, \mathbf{h})$ is structurally observable) then $\tilde{u}$ is connected to ($\tilde{y}$ is connected from) every vertex in $\mathcal{X}$. This coupled with the fact that $\mathbf{B}$ is rank-one gives us the following result.

Lemma 6: Under the condition that $(\mathbf{A}, \mathbf{c})$ is structurally controllable or $(\mathbf{A}, \mathbf{h})$ is structurally observable, there exists a $k \leq n$ such that $k \in I$.

Proof: First assume that $(\mathbf{A}, \mathbf{c})$ is structurally controllable. Then the generic rank of $\begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \cdots & \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix}$ equals n. This is equivalent to saying that for any structured row vector $\mathbf{h}^{\mathrm{T}}$ such that $\mathbf{h} \not\equiv \mathbf{0}$, $\mathbf{h}^{\mathrm{T}} \begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \cdots & \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix}$ is not zero identically. In other words, there exists at least one $k \in \{1, \ldots, n\}$ such that $\mathbf{h}^{\mathrm{T}}\mathbf{A}^{k-1}\mathbf{c} \not\equiv 0$, and hence $k \in I$. The proof for the situation when $(\mathbf{A}, \mathbf{h})$ is observable is almost identical. ■

This result also implies the fact that under the assumption of controllability and/or observability of the associated linear system, the set $\mathcal{W}_{\mathcal{X}_{\mathcal{Y}}\mathcal{U}}$ defined using the graph of the bilinear system is nonempty.

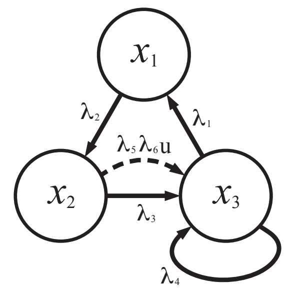

Fig. 2. A directed graph representation of a three-state bilinear system.

B. Proof of Main Result

Proof of 1 ⇔ 2*:* The structural controllability of the structured triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is equivalent to saying that the generic ranks of $\begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \cdots & \mathbf{A}^{n-1}\mathbf{c}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{h} & \mathbf{A}^{\mathrm{T}}\mathbf{h} & \cdots & (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{h}\end{bmatrix}$ both equal n, and the g.c.d. of all the elements of I equals 1. The generic rank conditions are equivalent to saying that $(\mathbf{A}, \mathbf{c})$ is structurally controllable and $(\mathbf{A}, \mathbf{h})$ is structurally observable. Using the results from [3], [4], alternatively one can say that the generic ranks of $\begin{bmatrix}\mathbf{A} & \mathbf{c}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{A}^{\mathrm{T}} & \mathbf{h}\end{bmatrix}^{\mathrm{T}}$ equal n and that $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are irreducible. Using Lemma 4, this is equivalent to the fact that the generic rank of $\mathbf{M}$ equals $n+1$ and that the irreducibility and g.c.d. conditions hold true.

Proof of 2 ⇔ 3*:* The first part of the equivalence follows from Lemma 4 and the results in [4], [6]. The second part of the equivalence follows from Lemma 5.

V. Examples

In order to help illustrate the important aspects of Theorem 1, consider the three-state bilinear system depicted in Fig. 2:

$$\mathbf{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & \lambda_4 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(t) + u(t) \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda_5 \end{bmatrix}}_{\mathbf{c}} \underbrace{\begin{bmatrix} 0 & \lambda_6 & 0 \end{bmatrix}}_{\mathbf{h}^{\mathrm{T}}} \mathbf{x}(t), \tag{5}$$

such that the parameters $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, 6$. As described earlier, we can define the associated linear system whose graphical representation is given by $G = (\mathcal{V}, \mathcal{E})$ such that the vertex set $\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}$ with $\mathcal{X} = \{x_1, x_2, x_3\}$, $\mathcal{U} = \{\tilde{u}\}$, and $\mathcal{Y} = \{\tilde{y}\}$. The edge set $\mathcal{E}$ is the union of the edge sets $\mathcal{E}_{\mathbf{A}} = \{(x_1, x_2), (x_2, x_3), (x_3, x_1), (x_3, x_3)\}$, $\mathcal{E}_{\mathbf{c}} = \{(\tilde{u}, x_3)\}$, and $\mathcal{E}_{\mathbf{h}} = \{(x_2, \tilde{y})\}$. The directed graph of the associated linear system is shown in Fig. 2. The edges belonging to $\mathcal{E}_{\mathbf{A}} \cup \mathcal{E}_{\mathbf{c}} \cup \mathcal{E}_{\mathbf{h}}$ are shown using solid lines. It can be easily seen that $\mathcal{X}, \mathcal{E}_{\mathbf{A}}$ is strongly connected (in fact a cyclic graph) with just one self-loop. As can be seen from the structures of $\mathbf{c}$ and $\mathbf{h}$, there is only one controlled edge in the network (the dashed line in Fig. 2). Recall that a controlled edge is the interconnection between nodes that be controlled via the external input to steer the system state trajectory.

First, note that the generic rank of $\mathbf{A}$ equals 3 and thus, $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{h}^{\mathrm{T}} & * \end{bmatrix}$ has full generic row rank. Furthermore,

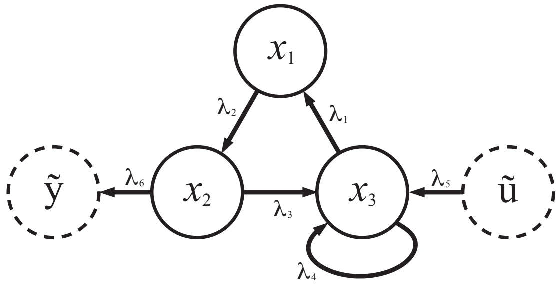

Fig. 3. The associated linear system corresponding to the bilinear system in Fig. 2.

both $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are irreducible which tells us that $(\mathbf{A}, \mathbf{c})$ is structurally controllable and $(\mathbf{A}, \mathbf{h})$ is structurally observable. Alternatively, one can deduce this from the directed graph G. For example, there exist paths $\tilde{u} \to x_3 \to x_1 \to x_2)$ and $(x_3 \to x_1 \to x_2 \to \tilde{y})$ which clearly tells us that all the conditions listed in part 3a of Theorem 1 are satisfied.

Similarly it can be observed that while $\mathbf{h}^{\mathrm{T}}\mathbf{c} = \mathbf{h}^{\mathrm{T}}\mathbf{A}\mathbf{c} = 0$, both $\mathbf{h}^{\mathrm{T}}\mathbf{A}^2\mathbf{c} = \lambda_1\lambda_2\lambda_5\lambda_6 \not\equiv 0$ and $\mathbf{h}^{\mathrm{T}}\mathbf{A}^3\mathbf{c} = \lambda_1\lambda_2\lambda_4\lambda_5\lambda_6 \not\equiv 0$. Thus, $3, 4 \in I$ and hence $\gcd(I) = 1$. Alternatively, this can be deduced from the fact that there exist walks $W_1 = \tilde{u} \to x_3 \to x_1 \to x_2$ of length 3 and $W_2 = \tilde{u} \to x_3 \to x_3 \to x_1 \to x_2$ of length 4, with $x_2 \in \mathcal{X}_{\mathcal{Y}}$, which are coprime. Since all the algebraic as well as graphical conditions listed in Theorem 1 are satisfied, the system described in (5) is structurally controllable. As an illustration, assume that $\lambda_1 = \cdots = \lambda_6 = 1$. The controllability and observability matrices for the associated linear system are as follows

$$\begin{bmatrix}\mathbf{c} & \mathbf{A}\mathbf{c} & \mathbf{A}^2\mathbf{c}\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}, \quad \begin{bmatrix}\mathbf{h}^{\mathrm{T}}\\ \mathbf{h}^{\mathrm{T}}\mathbf{A}\\ \mathbf{h}^{\mathrm{T}}\mathbf{A}^2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},$$

which clearly tells us that both have full rank and hence $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is controllable and observable. Furthermore, $\mathbf{h}^{\mathrm{T}}\mathbf{c} = \mathbf{h}^{\mathrm{T}}\mathbf{A}\mathbf{c} = 0$ whereas $\mathbf{h}^{\mathrm{T}}\mathbf{A}^2\mathbf{c} = \mathbf{h}^{\mathrm{T}}\mathbf{A}^3\mathbf{c} = 1$ implies that $3, 4 \in I$ and thus, $\gcd(I) = 1$. Thus, the triplet is controllable and the system (5) is structurally controllable.

Now suppose the self-loop (x_3, x_3) is removed. The resulting network is still strongly connected and the associated linear system is still both structurally controllable and observable. However, the set I now contains walks of lengths 3, 6, 9, and higher multiples of 3, which are not coprime to each other. Therefore, part 2b (or part 3b) of Theorem 1 fails and, the associated bilinear system is not structurally controllable.

VI. Conclusions

We formulated a set of easily verifiable algebraic and graph-theoretic conditions for the controllability of a class of structured bilinear systems. This class of single input systems with input matrices of rank one was motivated by the existing conventional controllability results for unstructured bilinear systems. We anticipate that we will be able to extend our results to the case of multiple input rank one bilinear systems. Removing the rank one limitation will be challenging due to the lack of exact algebraic conditions for testing controllability. However, we find that the insight drawn from the structural control criteria and the alternate graph-theoretic representation may help shed light on new perspectives in bilinear control, which could help the pursuit of a more general conventional (nonstructured) controllability result for bilinear systems.

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