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Control Configuration Design for a Class of Structural Bilinear Systems

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Abstract— **Analysis of structured systems has opened up the potential to understand the control properties of large-scale systems modeled as networks. Networks present novel questions to control theory because the interchangeability of nodes means that any subset could be controlled. In contrast, classic control systems have prescribed connections between controls and states. We present an algorithm to determine the structure of the input connectivity for single-input rank-one structured bilinear systems, which is analogous to designing the control configuration, or placement, of controls on edges in the network. In particular, we develop the control configuration with a minimum number of new interconnections (with respect to a cacti representation of the graph). These controls become the weights of certain edges in the network representation of the bilinear system.**

I. Introduction

The notion of structured control systems was developed so that the fundamental properties of controllability and observability could be generalized to system forms, or structure, rather than specific instances of that structure [1]. For example two machines may have the same parts and same assembly, but may differ by parameter values (stiffnesses, spring constants, drag coefficients), however, we would like to speak about the properties of this family of systems rather than each individual member in the family in isolation.

More recent work has leveraged structural control theory to study large scale systems, modeled as networks [2], [3]. These studies were motivated by the computational advantages offered by structural analysis over classic control theory. Real-world networks often have thousands or millions of nodes (states) and computing the Kalman rank condition on a linear system of that size is not scalable. At the same time, many real-world systems are modeled as networks because we lack the information to characterize their full dynamics. Even the weights that give the proportionality of connection between states is often unknown, therefore, tools like structural control are needed in order to study the controllability of systems in the presence of such ambiguity in parameter values.

The overwheliming majority of the work on structured systems has focused on linear systems [1], [4]–[7]. In the context of real-world networks, linear system models assume that exogenous controls manipulate the system. However, in some applications, it is not realistic to assume that we can have direct influence over the state of a given node; however, we may have the ability to manipulate the rate at which the nodes interact. For example, in cellular biochemical networks, we are largely unable to directly change the concentrations of protein within the cell, since medical drugs typically target receptors on the cell wall, which instead up- or down-regulate the expression of proteins in the cell. Importantly, we are effecting the rate at which processes happen, rather than directly effecting the states. A bilinear model of control permits us to capture this by effectively placing controls as the edge weights on some links in the network representation of the system.

Bilinear systems are a class of nonlinear systems that are jointly linear in the state and input dynamics. In other words, the system dynamics are linear in state for fixed input and linear in input for fixed states. Controllability and observability for classes of unstructured bilinear systems has been analyzed in for e.g., [8]–[13] - the most general and approachable of which involve the controllability of single-input, rank-one bilinear systems. Observability of structured bilinear systems was discussed in [14], [15], though note that because the nonlinearity of bilinear systems is induced by the control, observability of bilinear systems does not encounter the effects of the nonlinearity. Our work here and recently presented is the first discussion of structural controllability of bilinear systems [16]. In [16], given the sparsity patten of system matrices for a single input bilinear system with input matrix of rank one, we developed the algebraic and graph-theoretic conditions required for structural controllability. We showed that structural controllability of bilinear systems depends on two conditions: structural controllability and observability of an associated linear system obtained from a decomposition of the original system and the existence of walks of certain lengths in the associated system graph.

In this paper we will answer the converse question of control configuration design: to determine the minimal set of controlled edges to be added to a network to guarantee structural controllability, respecting the single-input rank-one constraint. Because most classic control problems have readily known input locations (e.g., lever points, actuators) the notion of control configuration design is trivial in these cases. The recent work with large-scale networks has introduced the importance of control configuration design in linear networks by selecting the nodes in the network to recieve exogenous input [2], [3]. We present the necessary background and theoretical extensions to build an algorithm for control configuration design for single-input rank-one bilinear systems.

II. Background

The system under investigation is a bilinear discrete-time system with a single input. Mathematically speaking, we

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focus our attention on the following class of systems:

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + u(t)\mathbf{B}\mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ represent the state and the external control input to the system, respectively. The matrices $\mathbf{A} = \left[a_{ji}\right], \mathbf{B} = \left[b_{ji}\right] \in \mathbb{R}^{n\times n}$ are structured matrices; i.e., their entries are either fixed zeroes or free (independent) parameters, which we denote in this paper using $*$ [1], [4], [16]. The objective in this paper is to design a structured matrix $\mathbf{B}$ for a given $\mathbf{A}$ with a minimum number of nonzero entries such that the pair $(\mathbf{A}, \mathbf{B})$ is structurally controllable. As in [8], [9], [12], [16], the matrix $\mathbf{B}$ is constrained to have a generic rank of at most one. The rationale for this constraint is simply that suitable algebraic conditions do not exist for any more general systems. However, we do observe that many meaningful and practical applications can be captured by such a form. Moreover, we anticipate that studying the structural controllability of single-input bilinear systems will help to lend deeper insight into the non-structured controllability of multi-input bilinear systems, which would encompass an even broader class of applications. In the rank-one case, $\mathbf{B}$ can be expressed as the product $\mathbf{c}\mathbf{h}^{\mathrm{T}}$ where $\mathbf{c}, \mathbf{h} \in \mathbb{R}^n$, so the state equation (1) becomes

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + u(t)\mathbf{c}\mathbf{h}^{\mathrm{T}}\mathbf{x}(t). \tag{2}$$

Structured systems naturally admit an alternative representation as directed graphs. Furthermore, representing systems such as (2) as graphs often leads to more intuitive explanations, algorithms, and proofs for properties of such systems. The structured matrix $\mathbf{A}$ defines a directed graph with n nodes and where the interconnections between the nodes are given by the matrix sparsity pattern. We will call $G(A) = (\mathcal{V}_{\mathbf{A}}, \mathcal{E}_{\mathbf{A}})$, with vertex set $\mathcal{V}_{\mathbf{A}} = \{x_1, \ldots, x_n\}$, the directed graph defined by the structured matrix $\mathbf{A}$. Given the edge set $\mathcal{E}_{\mathbf{A}}$, having an interconnection from node x_i to x_j is equivalent to $(x_i, x_j) \in \mathcal{E}_{\mathbf{A}}$ or $a_{ji} \not\equiv 0$ (note we use the $\not\equiv$ in the structural sense that, in this case, a_{ji} is not a fixed zero). In a specific realization of parameter values (non-structured matrices) the values a_{ji} represent the edge weights of the graph from node x_i to x_j. Note that the graph $G(\mathbf{A}^{\mathrm{T}})$ is obtained from $G(\mathbf{A})$ by simply reversing the direction of edges. The matrix $\mathbf{B}$ denotes the presence of *controlled edges* in the network, which represent links whose edge weights can be controlled over time. In a specific realization of parameter values, a non-zero entry b_{ji} implies an edge weight value of $b_{ji}u(t) = c_j h_i u(t)$ from node x_i to x_j. The rank-one constraint on $\mathbf{B}$ implies that these controlled edges have some special structure. In the simplest case, this could take the form of controlled edges leaving only from one node or entering only one node, however, the rank-one constraint can take on a variety of other forms. Therefore, in terms of the graphical interpretation of (2), our goal is to select the fewest number of controlled edges in the network so that the overall system becomes controllable.

A. Structural Controllability of Bilinear Systems

The objective of this paper is to design structured vectors $\mathbf{c}$ and $\mathbf{h}$ with a minimum number of non-zero entries for a given $\mathbf{A}$ such that the structured triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is controllable.

Definition 1: A system of the form (2) described by $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ is said to be structurally equivalent to another triple $(\tilde{\mathbf{A}}, \tilde{\mathbf{c}}, \tilde{\mathbf{h}})$ if there is a one-to-one correspondence between the locations of zeroes and free parameters of $\mathbf{A}$ and $\tilde{\mathbf{A}}$, $\mathbf{c}$ and $\tilde{\mathbf{c}}$, and $\mathbf{h}$ and $\tilde{\mathbf{h}}$, respectively.

Definition 2: A structured system $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ described by (2) is said to be structurally controllable if there exists a triplet $(\tilde{\mathbf{A}}, \tilde{\mathbf{c}}, \tilde{\mathbf{h}})$ that is structurally equivalent to $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ and is controllable.

The result for structural controllability of rank-one single-input discrete-time bilinear systems can be presented both in terms of structured linear algebra and in terms of graph theory. Reducibility effectively checks to ensure that all nodes can be reached from the controls, either directly or indirectly.

Definition 3: A structured matrix pair $(\mathbf{A}, \mathbf{B})$ is said to be reducible if and only if there exists a permutation matrix $\mathbf{P}$ such that $(\mathbf{A}, \mathbf{B})$ satisfies:

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{P}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}.$$

Using a decomposition technique similar to the one stated in [8], we define a linear system *associated* to the bilinear system, which is most intuitively understood as a graph. The overall augmented directed graph of the system described by (2) using the triplet $(\mathbf{A}, \mathbf{c}, \mathbf{h})$ can be defined as follows: the vertex set $\mathcal{V} = \mathcal{V}_{\mathbf{A}} \cup \{\tilde{u}\} \cup \{\tilde{y}\}$ where $\tilde{u}$ (the pseudo-input) and $\tilde{y}$ (the pseudo-observation) are the vertices whose connections to and from $\mathcal{V}_{\mathbf{A}}$ are given by $\mathbf{c}$ and $\mathbf{h}$, respectively. The edge set is the union of three sets: $\mathcal{E} = \mathcal{E}_{\mathbf{A}} \cup \mathcal{E}_{\mathbf{c}} \cup \mathcal{E}_{\mathbf{h}}$, such that $(\tilde{u}, x_j) \in \mathcal{E}_{\mathbf{c}}$ if and only if $c_j \not\equiv 0$, where c_j denotes the j^{th} entry of $\mathbf{c}$, and $(x_i, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}$ if and only if $h_i \not\equiv 0$, where h_i denotes the i^{th} entry of $\mathbf{h}$. In bilinear systems, controllability requires the simultaneous controllability and observability of the associated linear system. The pseudo-input and pseudo-observation are the two "halves" of each controlled edge: $h_i \not\equiv 0$ and $c_j \not\equiv 0$ implies that there is an edge from x_i to $\tilde{y}$ and from $\tilde{u}$ to x_j, respectively, but also means that there is a controlled edge in the bilinear system from x_i to x_j.

A walk W in a graph is a sequence of nodes and edges such that the begin vertex of an edge is the end vertex of the edge preceding it. If none of the vertices are repeated, the walk is called a path, otherwise in general vertices may be repeated in a walk. The number of edges in a walk (path) is called its length. A walk (path) is said to be *closed* if its begin and end vertices are the same. Two walks (paths) are called *disjoint* if there vertex sets are disjoint. A closed path is called a *cycle*. Paths, walks, and cycles can be written as either $x_0 \to x_1 \to \cdots \to x_k$ or $(x_0, x_1, \ldots, x_k)$. A path is called $\tilde{u}$-rooted ($\tilde{y}$-topped) if $\tilde{u}$ ($\tilde{y}$) is the begin (end) vertex of the path. A collection of mutually disjoint $\tilde{u}$-rooted ($\tilde{y}$-

topped) paths is called a $\tilde{u}$-rooted ($\tilde{y}$-topped) path family. Also a walk or a collection of walks is said to *cover* all the vertices in a set S if each of those vertices are a part of the walk or the collection of walks. Let $\mathcal{V}_{\tilde{y}} \subset \mathcal{V}_{\mathbf{A}}$ denote the set of nodes which have a direct edge to $\tilde{y}$; i.e., $\mathcal{V}_{\tilde{y}} = \{x_i \in \mathcal{V}_{\mathbf{A}} \colon (x_i, \tilde{y}) \in \mathcal{E}_{\mathbf{h}}\}$. Denote by $\mathcal{W}_{\mathcal{V}_{\tilde{y}}\tilde{u}}$ the set of all walks from $\tilde{u}$ to all the nodes in $\mathcal{V}_{\tilde{y}}$.

Lemma 1: [16] Consider the bilinear system (2) described by $(\mathbf{A}, \mathbf{c}, \mathbf{h})$. The following are equivalent:

1) The system (2) is structurally controllable.
2) (a) The generic rank of $\begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{h}^{\mathrm{T}} & * \end{bmatrix}$ equals $n+1$ and $(\mathbf{A}, \mathbf{c})$ and $(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are both irreducible.
   (b) The greatest common divisor of I equals one where $I = \{j \colon \mathbf{h}^{\mathrm{T}}\mathbf{A}^{j-1}\mathbf{c} \not\equiv 0, j = 1, \ldots, n^2\}$.
3) (a) In G the set of all the paths from $\tilde{u}$ to $\tilde{y}$ cover all the vertices in $\mathcal{V}_{\mathbf{A}}$ and there exists a disjoint union of $\tilde{u}$-rooted and correspondingly $\tilde{y}$-topped path family and cycle family that covers all the state vertices.
   (b) There exists a collection of walks of coprime lengths in $\mathcal{W}_{\mathcal{V}_{\tilde{y}}\tilde{u}}$.

Therefore, in either case, three conditions must be satisfied. The rank condition is equivalent to requiring only a single control, as we will see in the next section, so it is relatively trivially satisfied in the single-input case. The second is the reducibility condition, or that all nodes must be reached by the input and can reach the observation, since without such visibility from/to the input/output there is no hope for controllability. Finally the gcd condition must be satisfied, that at least two coprime walks must be found in the network.

III. Theory

This section contains the additional theory required to compose the algorithm to find a minimum number of entries in $\mathbf{B}$ for (2) to be rendered controllable.

The cactus is a specialized graph concept used to concisely describe (and find) the $\tilde{u}$-rooted and $\tilde{y}$-topped path/cycle families that cover the state vertices. A *stem* is a path in the graph and a *bud* is an cycle with a *distinguished edge* that connects a node of a stem to a node in the cycle. A *cactus* of the graph is composed of at most one stem and the attached buds and any number of disjoint (non-bud) cycles. A *cacti* is a collection of mutually disjoint cactus subgraphs. Importantly, cacti can be obtained using a maximum matching algorithm, which can be solved in polynomial time [2], [17]. Although the maximum matching result is not unique, it can be used as an efficient method to construct a family of paths and cycles from which we can then construct a minimum number of connections to the control and observation, thereby creating the $\tilde{u}$-rooted and $\tilde{y}$-topped path/cycle families required for controllability. In fact, part 3a of Lemma 1 is equivalent to saying that the augmented graphs for $G(\mathbf{A}, \mathbf{c})$ and $G(\mathbf{A}^{\mathrm{T}}, \mathbf{h})$ are each spanned by a cactus.

As we will see later, the number of controlled edges we add will be minimal with respect to the particular realization of the maximum matching (equivalently the particular cacti formed from the maximum matching). Because the maximum matching (cacti) is not unique, it may be possible to find a matching (cacti) that requires fewer controlled edges than another. As will be noted later, this will depend on the number and location of cycles created in the matching (cacti).

A graph G is said to possess a dilation if for a subset S of vertices, the set $T(S)$ of vertices incident to any of the elements in S is such that $|T(S)| < |S|$. We now define the concept of an atomic dilation which helps us locate the nodes which need to be connected to controls.

Definition 4: A set $S \subset \mathcal{V}_{\mathbf{A}}$ is said to possess an atomic dilation if S contains a dilation, but no proper subset of S contains a dilation.

It is stated in [18] that a dilation in a graph is equivalent to a rank deficiency in the structured adjacency matrix of the graph. We present a proof of a related statement which relates the rank deficiency to the existence and unicity of atomic dilations.

Lemma 2: For each atomic dilation set S, we have $|T(S)| = |S| - 1$.

Proof: Assume S contains an atomic dilation. By definition, then, $|T(S)| < |S|$. Assume $|T(S)| \le |S| - 2$. Note that $T(S)$ contains all the nodes which have edges incident to any node in S. Now consider a new set $\tilde{S}$ formed by removing a node (any node) from S so that $|\tilde{S}| = |S| - 1$. Let $T(\tilde{S})$ be the corresponding set for $\tilde{S}$. Since $\tilde{S} \subset S$, we have $T(\tilde{S}) \subseteq T(S)$, and thus $|T(\tilde{S})| \le |T(S)| \le |S| - 2 < |\tilde{S}|$. Thus, $\tilde{S}$ contains a dilation, which is a contradiction. ■

In prior work, our proof indicates that for the system in (2) to be controllable, the structured matrix $\mathbf{A}$ must be of at least rank $n-1$ [16]. We use this fact now, which is due to the imposed single-input rank-one constraint, to establish a connection between the rank deficiency of $\mathbf{A}$ and dilations. In the following, we will denote by $\mathbf{A}|_S$ the set of rows of corresponding to the indices in $S \subseteq \mathcal{V}_{\mathbf{A}}$.

Lemma 3: The generic rank of an $n \times n$ structured matrix $\mathbf{A}$ equals $n-1$ if and only if the associated directed graph $G(\mathbf{A})$ has a unique atomic dilation.

Proof: We will only present a sketch of the proof. First, assume that $G(\mathbf{A})$ contains a unique atomic dilation set $S \subseteq \mathcal{V}_{\mathbf{A}}$. Using Lemma 2, one can then conclude that $\mathbf{A}|_S$ has at most $|S|-1$ nonzero columns and thus, a generic rank of at most $|S|-1$. If S is a singleton, then $\mathbf{A}|_S = \mathbf{0}$, and this immediately proves that the generic rank of $\mathbf{A}|_S = 0 = |S| - 1$. Assume S contains more than one element and consider a set $\{x_i\} \subset S$. Since $\{x_i\}$ does not possess a dilation, the x_i^{th} row must have at least one nonzero entry. Since x_i was arbitrary, every row in S contains at least one nonzero entry, and $\mathbf{A}|_S$ contains at least one nonzero column. Since the number of nonzero columns of $\mathbf{A}|_S$ is bounded from above by $|S|-1$ and below by 1, if $|S| = 2$, $\mathbf{A}|_S$ must have exactly one nonzero column and, therefore, possesses a generic rank of one. However, if $|S| > 2$, one can then consider a proper subset $\{x_i, x_j\} \subset S$ such that $x_i, x_j \in S$. Since this set does not contain a dilation, there will be at least two nonzero columns in $\mathbf{A}|_{\{x_i,x_j\}}$ in addition to the fact that the x_i^{th} and x_j^{th} rows contain at least one nonzero entry each. This implies that the generic rank of $\mathbf{A}|_{\{x_i,x_j\}}$ equals 2. An induction argument then gives us that the generic rank of every subset R of S having $|S|-1$ elements is exactly $|S|-1$. Thus, the generic rank of $\mathbf{A}|_S$ is equal to $|S|-1$. Using a similar argument, one can then deduce that the generic rank of $\mathbf{A}|_{\mathcal{V}_\mathbf{A}\setminus S}$ equals $|\mathcal{V}_\mathbf{A}\setminus S|$. Also, any proper subset of rows of $\mathbf{A}|_S$ is linearly independent to the rows in $\mathbf{A}|_{\mathcal{V}_\mathbf{A}\setminus S}$ (otherwise that will contradict with the unicity of S). This implies that the generic rank of $\mathbf{A}$ is $n-1$.

On the other hand, assume that the generic rank of $\mathbf{A}$ is $n-1$. Identify the smallest set $S \subseteq \mathcal{V}_\mathbf{A}$ such that $\mathbf{A}|_S$ contains linearly dependent rows but the rows in any of the proper subsets of S are linearly independent. If $S=\{x_i\}$ for some $x_i \in \mathcal{V}_\mathbf{A}$, then obviously $\mathbf{A}|_S=\mathbf{0}$ and hence, S is an atomic dilation. Else, consider a set $\{x_i\}\subset S$. Since, $\mathbf{A}|_{\{x_i\}}$ is linearly independent, the x_i^{th} row contains at least one nonzero entry. Again an induction argument shows that no proper subset of S contains a dilation. In particular every subset $R\subset S$ of $|S|-1$ entries has at least $|S|-1$ nonzero columns. In other words, $|T(R)|\geq|R|\geq|S|-1$. This coupled with the fact that $\mathbf{A}|_S$ contains a set of linearly dependent rows then gives us that $|T(S)|=|S|-1$. So, S possesses an atomic dilation. One can then use a set-theoretic argument to show that S is unique since if there are two atomic dilations S and T, then $S\cup T$ will have two atomic dilations and correspondingly the generic rank of $\mathbf{A}$ would be at most $n-2$ leading to a contradiction.

■

Remark 1: When the graph has a single dilation, we also observe that there can be only one cactus (i.e., a single $\tilde{u}$-rooted path/cycle family covers all the vertices and a single $\tilde{y}$-topped path/cycle family covers all the vertices) [1], [4], [5]. This implies that in a graph with one dilation, there is at most one stem with any number of corresponding buds and cycles. The implication does not go the other way, however, because if $\mathbf{A}$ has full generic rank and there is no dilation in $G(\mathbf{A})$, there is still considered to be one cactus, but composed entirely of cycles (i.e., no stem or buds).

IV. Algorithm

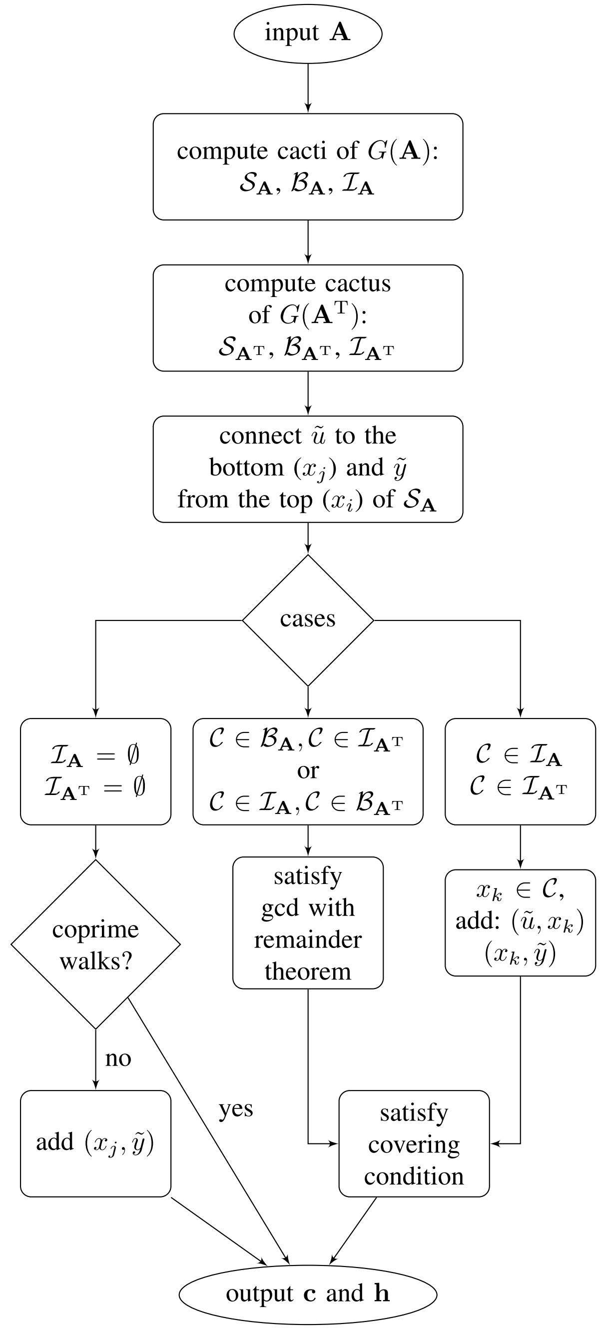

Fig. 1. A high level flow chart of the algorithm.

Using the established background in Section II and the new observations made in Section III, we now synthesize an algorithm to design the control configuration for the bilinear system (2) to render it controllable. The flowchart in Fig. 1 depicts the high-level sequence of the algorithm. The input to the algorithm is $\mathbf{A}$ and the outputs are $\mathbf{c}$ and $\mathbf{h}$. The entries of the vectors $\mathbf{c}$ and $\mathbf{h}$ are initially set equal to zero. Recall that connecting $\tilde{u}$ to x_j implies setting c_j to be a free parameter; i.e., $c_j \not\equiv 0$. Similarly, connecting x_i to $\tilde{y}$ is equivalent to making h_i a free parameter.

1) **Build the cacti of $G(\mathbf{A})$:** Using a maximum matching algorithm, generate the cacti of $G(\mathbf{A})$. If more than one cactus is created, by Remark 1 this implies that the rank of $\mathbf{A}$ is less than $n-1$. As mentioned previously, the generic rank of $\mathbf{A}$ must be at least $n-1$ for a rank-one single-input to potentially make the system controllable. Therefore, only continue if there is a single cactus. As explained earlier, the cactus will have a stem with corresponding buds, and/or isolated cycles. Let $\mathcal{S}_\mathbf{A}$ denote the stem of the cacti, $\mathcal{B}_\mathbf{A}$ be collection of buds, and $\mathcal{I}_\mathbf{A}$ be the collection of isolated cycles.
2) **Build the cactus of $G(\mathbf{A}^\mathrm{T})$:** Obtain the cactus of $G(\mathbf{A}^\mathrm{T})$ from the cactus of $G(\mathbf{A})$. Although we could generate the cactus of $G(\mathbf{A}^\mathrm{T})$ directly using the maximum matching algorithm on the graph with edges reversed, the following method uses the fact that there is a single cactus for $G(\mathbf{A})$ and for $G(\mathbf{A}^\mathrm{T})$ in order to maintain the original buds and isolated cycles: $\mathcal{B}_\mathbf{A}\cup$

$\mathcal{I}_{\mathbf{A}} = \mathcal{B}_{\mathbf{A}^{\mathrm{T}}} \cup \mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$.

- **Identify the stem:** Reverse the direction of edges in the stem to obtain the stem $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$.
- **For each cycle, determine if it is a bud or cycle:** For each cycle $\mathcal{C}$ in $\mathcal{B}_{\mathbf{A}}$ and $\mathcal{I}_{\mathbf{A}}$, check whether there exists a distinguished edge, in $G(\mathbf{A}^{\mathrm{T}})$, from any node in $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$ to any node in $\mathcal{C}$. If a distinguished edge exists, $\mathcal{C} \in \mathcal{B}_{\mathbf{A}^{\mathrm{T}}}$, i.e., $\mathcal{C}$ is a bud in the cacti of $G(\mathbf{A}^{\mathrm{T}})$. If there is no distinguished edge $\mathcal{C} \in \mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$, i.e., $\mathcal{C}$ is an isolated cycle in the cacti of $G(\mathbf{A}^{\mathrm{T}})$.

3) **Connect controls:** If the cactus of $G(\mathbf{A})$ has a stem, let its length be $\ell - 1$ and connect $\tilde{u}$ to the base node (say x_j) and $\tilde{y}$ from the top node (say x_i) of the stem, i.e., set c_j and h_i to be free parameters. Proceed according to one of the following cases:

I **No isolated cycles:** Since all nodes are contained in the stem and/or the buds connected from the stem, connecting $\tilde{u}$ and $\tilde{y}$ as above guarantees that all nodes are reachable from $\tilde{u}$ and can reach $\tilde{y}$. To check the coprime walk condition, let $I = \{k \colon a_{ij}^{k-1}, \ell = 1, \dots, n^2\}$ where a_{ij}^{k-1} denotes the (i,j) entry of $\mathbf{A}^{k-1}$. If $\gcd(I) = 1$, then the condition is naturally satisfied. Otherwise add an edge from the bottom node of the stem, x_j, to $\tilde{y}$. This inserts a controlled self-loop at x_j: $\tilde{u} \to x_j \to \tilde{y}$, which automatically satisfies the coprime walks since the $\gcd(\ell, 1) = 1$ (ℓ is the length of the walk from $\tilde{u}$ to the neighbor set of $\tilde{y}$, $\mathcal{V}_{\tilde{y}}$). Note that this procedure also addresses the case when the cactus only has a stem and no buds.

II **Same isolated cycle in both cacti:** If there is at least one cycle that is isolated in both cacti, i.e., $\mathcal{C} \in \mathcal{I}_{\mathbf{A}}$ and $\mathcal{C} \in \mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$, choose any node (say x_k) of such a cycle and connect $\tilde{u}$ to x_k and x_k to $\tilde{y}$. This adds a controlled self-loop in the system which again immediately satisfies the gcd condition. To satisfy the covering condition, also connect $\tilde{u}$ to any node in each of the rest of the isolated cycles $\mathcal{I}_{\mathbf{A}}$ and connect $\tilde{y}$ to any node in each of the rest of the isolated cycles in $\mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$. Note that this procedure also deals with the case when the cactus does not have a stem, in which case all the cycles are the same in both cacti.

III **Bud becomes isolated cycle:** Consider first the case where the cycle is a bud in the cactus of $G(\mathbf{A})$, $\mathcal{C} \in \mathcal{B}_{\mathbf{A}}$, but is an isolated cycle in the cactus of $G(\mathbf{A}^{\mathrm{T}})$, $\mathcal{C} \in \mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$. Let $k \leq \ell$ be the distance from $\tilde{u}$ to the end vertex (say x_p) of the distinguished edge connecting the stem $\mathcal{S}_{\mathbf{A}}$ to $\mathcal{C}$ and c be the cycle length.

Obtain the smallest remainder r such that $|\ell - 1 - k| = q \cdot c + r$. Starting with x_p as the 0^{th} vertex, connect the r^{th} vertex in $\mathcal{C}$ to $\tilde{y}$. Note that $\gcd(\ell, k + (q \cdot c + r)) = \gcd(\ell, \ell \pm 1)$, where the length $k + (q \cdot c + r)$ is attained from the walk from $\tilde{u}$ to x_p (k), around the bud q times ($q \cdot c$), and partially around again to the r^{th} node (r), which is part of the neighbor set of $\tilde{y}$, $\mathcal{V}_{\tilde{y}}$.

To satisfy the covering condition, also connect $\tilde{u}$ to any node in each of the rest of the isolated cycles $\mathcal{I}_{\mathbf{A}}$ and connect $\tilde{y}$ to any node in each of the rest of the isolated cycles in $\mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$.

The case where $\mathcal{C}$ is a bud in the cactus of $G(\mathbf{A}^{\mathrm{T}})$, i.e., $\mathcal{C} \in \mathcal{B}_{\mathbf{A}^{\mathrm{T}}}$ and a cycle in the cactus of $G(\mathbf{A}^{\mathrm{T}})$, i.e., $\mathcal{C} \in \mathcal{I}_{\mathbf{A}}$ is treated identically, however, the roles are reversed of $\tilde{u}$ and $\tilde{y}$.

Our goal in designing the vectors $\mathbf{c}$ and $\mathbf{h}$ is to satisfy the controllability and observability of the linear system and the gcd condition. The former can be done by connecting $\tilde{u}$ to ($\tilde{y}$ from) the isolated cycles in the cacti and the dilation, which is the bottom (or top) of the stem. Assume the number of cacti components; i.e., stem (if present) plus the number of isolated cycles in $G(\mathbf{A})$ is equal to $n_{\mathbf{c}}$. Similarly, let the number of cacti components in $G(\mathbf{A}^{\mathrm{T}})$ be $n_{\mathbf{h}}$. Hence, at minimum $n_{\mathbf{c}}$ edges from $\tilde{u}$ and $n_{\mathbf{h}}$ edges to $\tilde{y}$ are needed to guarantee controllability and observability of the associated linear system. However, due to the rank-one constraint on $\mathbf{B}$, it is also evident that $n_{\mathbf{c}} n_{\mathbf{h}}$ is the minimum number of controlled edges (or equivalently, the number of non-zero entries in $\mathbf{B}$) required to guarantee overall controllability of the bilinear system. What we find is that bilinear systems are relatively easy to control, in the sense that most of the cases above attain this minimum even when the gcd condition is tested. The exception is Case I, in which only a single extra controlled edge (a self edge) may be required to satisfy the coprime walks. Note that for Case I the number of required controlled edges is either 1 or 2, so even in this case the gcd condition does not cause significant effect on the control configuration.

Remark 2: It was mentioned briefly in the preceding section that the maximum matching algorithm does not yield a unique solution, hence the the cactus found in the first step of our algorithm has some degeneracy [2], [17]. In particular, the numbers $n_{\mathbf{c}}$ and $n_{\mathbf{h}}$ depend on the number of isolated cycles in the cactus and, therefore, the number of controlled edges placed in the network depends on the structure of the resulting cactus. While the problem of obtaining the cacti for a directed graph using maximum matching is solvable in polynomial time, the problem of enumerating all the potential cacti of a given graph is known to be NP-hard - in particular, combinatorial in the number of dilations. Ideally we would like a maximum matching algorithm that returns the fewest number of isolated cycles, since this will make $n_{\mathbf{c}}$ and $n_{\mathbf{h}}$ smaller. Fortunately, Lemma 3 indicates that there is a single, unique dilation in this context, so enumerating the matchings is tractable. However, it is a worthy point to make for extensions of this work that lift the rank restriction on $\mathbf{A}$.

Remark 3: We have outlined the procedure for assigning controlled edges for the simplest types of cacti, in which buds only occur off of (with distinguished edges coming from) the stem. In general, however, it is possible for buds to occur off of other cycles - both buds and isolated cycles - and for there to be chains of such buds off of buds. The placement of controlled edges in these cases follows the same logic - and yields the same results - as we have already described, however, the arithmetic and notation becomes

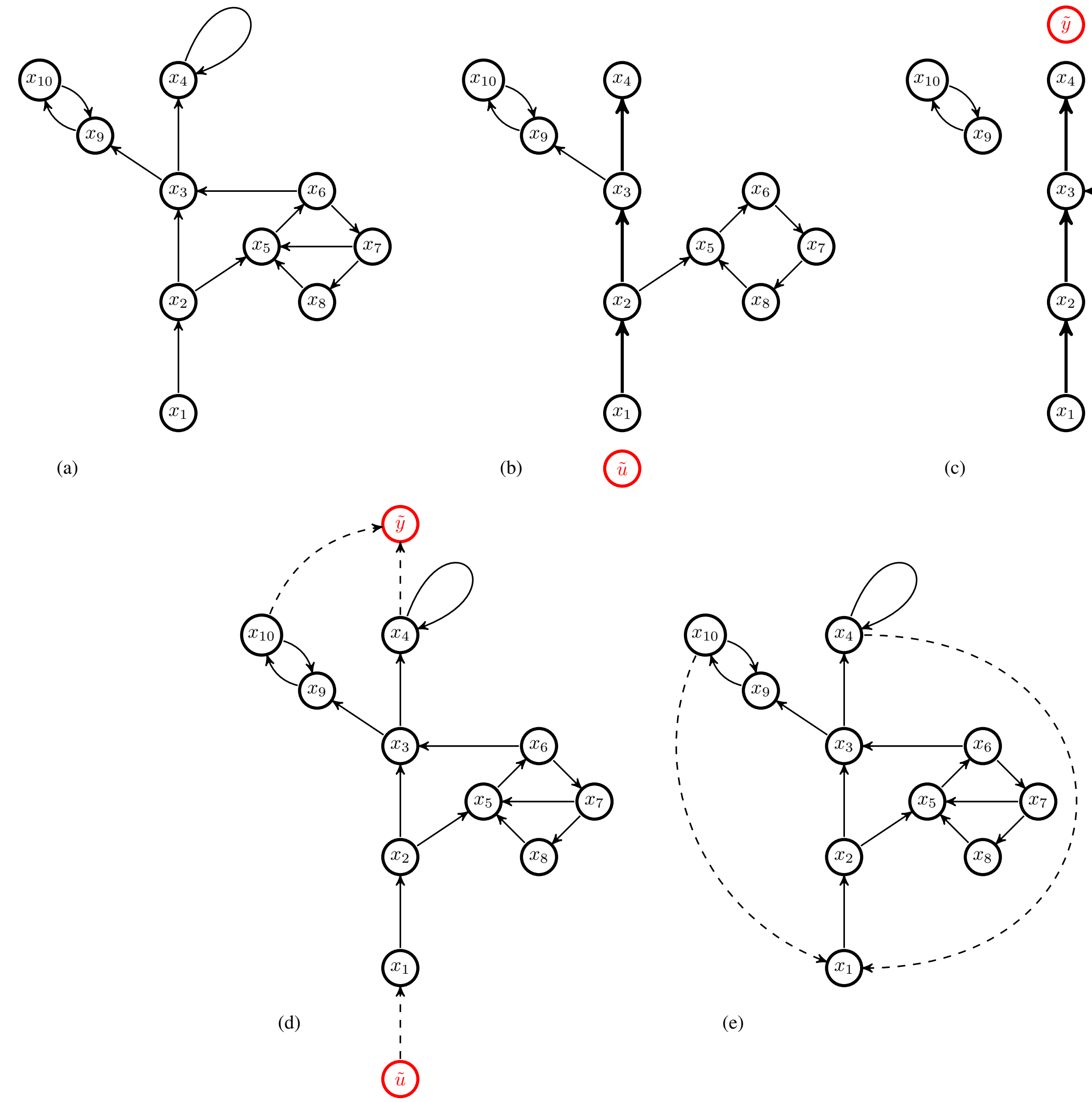

Fig. 2. The 10 node graph discussed in Example 1, showing (a) the original graph (b) the cactus of $G(\mathbf{A})$ with stem $\mathcal{S}_{\mathbf{A}}$ bolded (c) the cactus of $G(\mathbf{A}^{\mathrm{T}})$ with stem $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$ bolded (d) the resulting control augmented graph of the associated linear system (e) the resulting control augmented graph of the original system. In all diagrams, dashed arrows indicate edges added to the graph.

more cumbersome. In light of clarity and space constraints, we have omitted these details.

Remark 4: Self-loops can play a dramatically simplifying role in the control of bilinear systems. Once the conditions of controllability and observability of the associated linear system are satisfied, the presence of a self-loop (either using the existing or controlled interconnections) automatically satisfies the gcd condition. Suppose there exists a self-loop on node k; i.e., $a_{kk} \not\equiv 0$. The controllability and observability grant that a path exists from $\tilde{u}$ to $\tilde{y}$ in the augmented system graph via node k. Adding the self-loop to this path gives a walk of length one more than the original path, which clearly satisfies the gcd condition. For the situation where $b_{kk} = c_k h_k \not\equiv 0$, we have $1 \in I$ which satisfies the gcd condition automatically.

Remark 5: We briefly explore the computational complexity of the algorithm, in particular when checking the gcd condition is required (Case I). As mentioned in [17] the number of steps required for obtaining the cacti represenation via maximum matching is $O(n^{5/2})$. Obtaining cacti of $G(\mathbf{A}^{\mathrm{T}})$ from the cacti of $G(\mathbf{A})$ involves checking whether every cycle in the cactus for $G(\mathbf{A})$ is a bud or isolated cycle in the cactus for $G(\mathbf{A}^{\mathrm{T}})$, requiring at most $n^2/4$ operations. Checking for the lengths of stem and cycles involves $O(n)$ operations giving us a total complexity of $O(n^{5/2}+n^2+n) = O(n^{5/2})$. In Case I we also need to check for coprime walks, which effectively requires matrix multiplication of $\mathbf{A}$ up to the power n^2. The most efficient matrix multiplications run

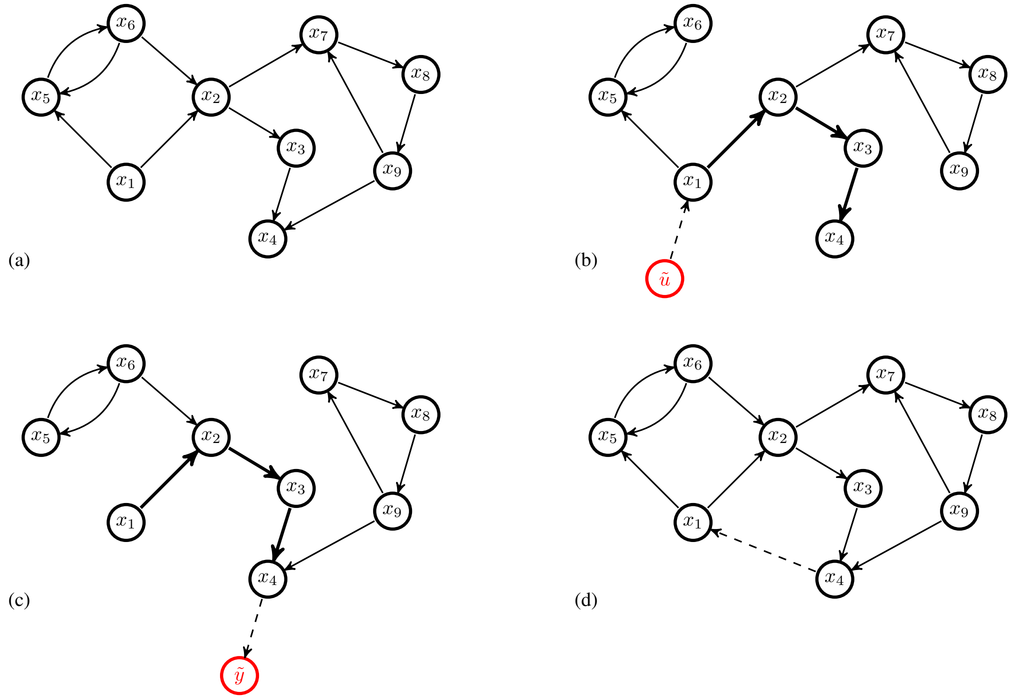

Fig. 3. The 9 node graph discussed in Example 2 showing (a) the original graph (b) the cactus of $G(\mathbf{A})$ with input connectivity shown (c) the cactus of $G(\mathbf{A}^{\mathrm{T}})$ with output connectivity shown (d) the resulting control augmented graph of the original system. In all diagrams, dashed arrows indicate edges added to the graph and bold arrows indicate the stem of a cactus.

in $O(n^{2.3})$, so computing these powers to n^2 is $O(n^{4.3})$. Remark 4 provides a convenient shortcut if self-loops exist in the network. Note that to guarantee the controllability of the bilinear system in Case I without checking the gcd condition, only one additional controlled edge is required. It may be worthwhile, then, to compare the cost of the computation time related to the gcd condition with the cost of inserting another controlled edge in the network.

V. Examples

In this section we will present two illustrative examples to demonstrate our algorithm.

A. Example I

Consider the 10 node bilinear network as shown in Fig. 2. Decomposing the graph $G(\mathbf{A})$ into its cacti using maximum matching yields the following elements: a stem $\mathcal{S}_{\mathbf{A}}$ comprising of the nodes (x_1, x_2, x_3, x_4) and two buds namely, $\mathcal{C}_1 = (x_5, x_6, x_7, x_8)$ and $\mathcal{C}_2 = (x_9, x_{10})$.

The cactus for $G(\mathbf{A}^{\mathrm{T}})$ is obtained from the cactus of $G(\mathbf{A})$. While the edge (x_2, x_5) connects the bud $\mathcal{C}_1$ to the stem $\mathcal{S}_{\mathbf{A}}$, a different distinguished edge (x_6, x_3) connects $\mathcal{C}_1$ to $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$. Therefore, the cycle $\mathcal{C}_1$ is a bud in both the cacti. The cycle $\mathcal{C}_2$, however, has no distinguished edge to $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}}$ and becomes an isolated cycle in the cacti for $G(\mathbf{A}^{\mathrm{T}})$. Hence, the cactus for $G(\mathbf{A}^{\mathrm{T}})$ contains one bud $\mathcal{C}_1 \in \mathcal{B}_{\mathbf{A}^{\mathrm{T}}}$ and one isolated cycle $\mathcal{C}_2 \in \mathcal{I}_{\mathbf{A}^{\mathrm{T}}}$ in addition to the stem $\mathcal{S}_{\mathbf{A}^{\mathrm{T}}} = (x_4, x_3, x_2, x_1)$.

This network is an example of Case III. Therefore, to control the linear system associated to $G(\mathbf{A})$, an edge must be added from the pseudo-input $\tilde{u}$ to x_1. The edge $(\tilde{u}, x_1)$ connects all the nodes from $\tilde{u}$ since there are no isolated cycles. To observe the linear system it is necessary to obtain observations from both the top of the stem $\mathcal{S}_{\mathbf{A}}$ (from node x_4) and from one of the nodes in $\mathcal{C}_2$. There will be two edges to $\tilde{y}$ and one to $\tilde{u}$, yielding two controlled edges.

There is ambiguity in which node in $\mathcal{C}_2$ is connected to $\tilde{y}$, however, we can use the gcd requirement to help make this choice. The length of the stem is 3, therefore, $\ell = 4$ is the distance from $\tilde{u}$ to $x_4 \in \mathcal{V}_{\tilde{y}}$ along the stem. The distance $k = 4$ is the length from $\tilde{u}$ to x_9 and $c = 2$ is the length of cycle $\mathcal{C}_2$. Therefore, $|\ell - 1 - k| = 1$, so $q = 0$ and $r = 1$ making $q \cdot c + r = \ell + 1 = 5$. Therefore, we add the controlled edge from $\mathcal{C}_2$ to $\tilde{y}$ at the r^{th} node of the cycle, x_{10}. The walk from $\tilde{u}$ to the $r = 1$ node of the cycle, $x_{10} \in \mathcal{V}_{\tilde{y}}$, is of length 5, which is coprime with $\ell = 4$. The bilinear network becomes controllable with just two controlled edges, namely (x_4, x_1) and (x_{10}, x_1).

B. Example II

Consider the 9 node network as shown in Fig. 3. The cactus of $G(\mathbf{A})$ is composed of a stem $\mathcal{S}_{\mathbf{A}} = (x_1, x_2, x_3, x_4)$ and two buds $\mathcal{C}_1 = (x_5, x_6)$ and $\mathcal{C}_2 = (x_7, x_8, x_9)$ attached to the stem by the edges (x_1, x_5) and (x_2, x_7). Similarly, in the cactus for $G(\mathbf{A}^{\mathrm{T}})$, the cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ are also connected to the stem by the edges (x_9, x_4) and (x_6, x_2), respectively.

Thus, a single edge $(\tilde{u}, x_1)$ achieves controllability for the associated linear system, and a single edge $(x_4, \tilde{y})$ guarantees observability of the associated linear system.

We now check the existence of coprime walks from $\tilde{u}$ to $\mathcal{V}_{\tilde{y}} = \{x_4\}$. Multiplying the powers of $\mathbf{A}$, we observe that $a_{41}^3 \neq 0$ and $a_{41}^8 \neq 0$. These correspond to the paths $(\tilde{u}, x_1, x_2, x_3, x_4)$ and $(\tilde{u}, x_1, x_2, x_7, x_8, x_9, x_7, x_8, x_9, x_4)$ which are of coprime lengths, 4 and 9 respectively. Thus, only one controlled edge (x_4, x_1) is sufficient to control the bilinear network. We note that if the cost of checking the coprime paths is prohibitive (e.g., for a larger network) we could add an extra controlled self-loop $(x_1, \tilde{y})$ to guarantee controllability without needing to check the gcd condition.

VI. Conclusions

We presented an algorithm to selectively add controlled edges to a bilinear network to render it structurally controllable. This process invovles decomposing a given network into a cactus, guaranteeing controllability and observability of the associated linear system, and guarantee the existence of coprime walks in the network. Here we consider single-input rank-one discrete time bilinear systems, however, we anticipate that the intuition gained from the structural analysis of controllability will help to generalize the known results for controllability of structured and non-structured bilinear systems.

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